IL
NUOVO CIMENTO
ORGANO DELLA SOCIETÀ ITALIANA DI FISICA
SOTTO GLI AUSPICI DEL CONSIGLIO NAZIONALE DELLE RICERCHE

VOL. XVII, N. 6 *Serie decima* 16 Settembre 1960

Sur une méthode simplifiée de calcul pour les processus relativistes en électrodynamique quantique.

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(ricevuto l'11 Aprile 1960)

Summary. — We propose a simplified method of approximate calculation for relativistic processes in quantum electrodynamics. Our starting point is a quantum field theoretical generalization of the semi-classical Williams-Weizsäcker method, which defines an equivalent photon spectrum for a relativistic charged particle. We then show that a formula of the Williams-Weizsäcker type can be set up to express the probability associated with any elementary diagram containing two fermion lines and one photon line, provided that both fermion states are relativistic. The elementary virtual processes represented by these diagrams may thus be considered as « quasi real processes ». Several examples are given, in which this concept is applied to calculate higher order processes by relating them directly to simpler effects. The cases we treat are: 1) bremsstrahlung of a relativistic particle at a given angle; 2) nuclear recoil from pair creation by high-energy photons; 3) inner bremsstrahlung in the disintegration of the μ-meson; 4) radiative corrections in the compared β-ray spectra of ^{12}B and ^{12}N. In conclusion, further possibilities to extend the method (especially to π-meson theory) are briefly discussed.

1. – Introduction.

La méthode de calcul que nous exposons ci-après nous a été inspirée essentiellement par deux considérations:

a) Le fait que les formules obtenues en électrodynamique quantique pour les sections efficaces des phénomènes de différents ordres se simplifient en général grandement lorsque l'on passe au cas extrême-relativiste.

b) Le succès obtenu, particulièrement pendant ces dernières années, par la méthode semi-classique élaborée par WILLIAMS et von WEIZSÄCKER vers 1933-35. On sait que cette méthode (fondée sur l'emploi d'un spectre équivalent de photons pour une particule chargée rapide) a permis d'interpréter de façon satisfaisante les résultats expérimentaux concernant les interactions nucléaires des mésons μ du rayonnement cosmique, et aussi certains effets nucléaires obtenus avec les électrons de haute énergie produits par des accélérateurs.

Nous en avons conclu qu'il devait être possible de développer une méthode assez générale visant à simplifier les calculs dans un grand nombre de problèmes où interviennent des particules relativistes.

Notre point de départ est une reformulation de la méthode de Williams et Weizsäcker, permettant de fonder celle-ci sur une base conforme à la théorie quantique des champs et d'aboutir à une formule d'une validité plus générale.

Nous allons montrer que le même type de formule se prête en fait à une utilisation beaucoup plus large que celle prévue par la méthode originelle. Le cas d'un spectre de photons virtuels devient ainsi un cas particulier, parmi de nombreux autres cas de l'électrodynamique que la nouvelle méthode permet de traiter.

Nous donnerons quelques applications où notre procédé a pour effet de simplifier la solution de divers problèmes d'un intérêt actuel. En conclusion, nous discuterons des possibilités de généralisation ultérieure.

2. – Généralisation de la méthode de Williams et Weizsäcker.

Le principe de la méthode d'approximation semiclassique de Williams et Weizsäcker [1,2] est de décomposer le champ électromagnétique créé par une particule chargée relativiste au voisinage d'une cible en un spectre équivalent de photons. Pour ceci, on suppose réalisées les deux conditions:

$$E \gg m\,, \qquad E \gg k\,, \tag{1}$$

où E et m sont l'énergie et la masse de la particule, k l'énergie cédée par elle dans l'interaction avec la cible (énergie du photon virtuel échangé). On obtient ainsi la section efficace pour un processus A dû à l'interaction électromagnétique de la particule, simplement en intégrant sur le spectre équivalent de photons la section efficace pour le processus analogue B provoqué par des

[1] E. J. WILLIAMS: *Proc. Roy. Soc.*, A **139**, 163 (1933); *Phys. Rev.*, **45**, 729 (1934); *Kgl. Dansk. Vid. Selsk.*, **13**, 4 (1935).

[2] C. F. VON WEIZSÄCKER: *Zeits. Phys.*, **88**, 612 (1934).

photons libres. Donc:

$$\sigma_A = \int P(k)\,\sigma_B\,\mathrm{d}k\,, \tag{2}$$

où le spectre de photons $P(k)$ est donné [3] par la formule:

$$P(k)\,\mathrm{d}k = \frac{2\alpha}{\pi}\,\frac{\mathrm{d}k}{k}\left(\ln\frac{E}{mkb_{\min}} - 0.4\right). \tag{3}$$

Ici, α est la constante de structure fine; b est le paramètre d'impact de la particule par rapport à la cible. (Nous prenons $\hbar = c = 1$, et nous supposons une charge $Z = 1$ pour la particule.) Il a été admis couramment que l'on pouvait choisir un paramètre d'impact de l'ordre de la longueur d'onde de Compton de la particule; dans ce cas, la formule devient:

$$P(k)\,\mathrm{d}k = \frac{2\alpha}{\pi}\,\frac{\mathrm{d}k}{k}\left(\ln\frac{E}{k} - 0.4\right). \tag{4}$$

Nous avons déjà demontré précédemment [4] que l'on obtient une formule tout à fait semblable, avec les mêmes hypothèses, à partir du formalisme de le théorie quantique des champs. Nous allons montrer ici que l'on peut généraliser la formule en renonçant à la seconde hypothèse qui figure dans (1).

Nous nous proposons de comparer, afin d'arriver à une expression du type (2), les diagrammes A et B de la Fig. 1, où nous attribuons la même énergie $k\,(= E - E')$ au photon libre de B et au photon virtuel de A; le diagramme restant M est le même dans les deux cas.

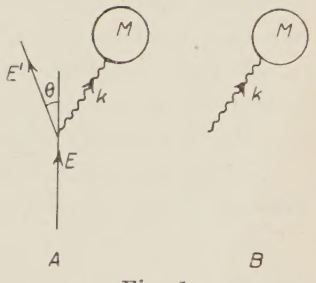

Fig. 1.

La seule condition que nous imposons est que la particule soit relativiste avant et après sa diffusion: $E, E' \gg m$.

Nous admettrons dans la suite que la particule est un fermion. On pourrait faire une démonstration analogue pour un boson. Mais il faut remarquer que la méthode concerne surtout les particules dont l'intéraction avec la matière est essentiellement électromagnétique, c'est-à-dire les électrons et les mésons μ.

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[3] W. Heitler: *The Quantum Theory of Radiation*, 3ème éd. (Oxford, 1954), p. 414 et suiv.

[4] D. Kessler et P. Kessler: *Nuovo Cimento*, **4**, 601 (1956).

En appliquant les règles de Feynman au calcul des diagrammes A et B, nous obtenons:

$$\sigma_A = \frac{\alpha}{2\pi^2}\int \frac{\mathrm{d}^3p'}{(\mathbf{p}-\mathbf{p}')^4}\,|\,\overline{w}'\boldsymbol{\gamma}w\cdot\mathbf{M}\,|^2\,, \tag{5}$$

$$\sigma_B = \frac{1}{2k}\,|\,\boldsymbol{\epsilon}\cdot\mathbf{M}\,|^2\,, \tag{6}$$

où: $\mathbf{p}$, $\mathbf{p}'$ sont les quadrivecteurs impulsion-énergie pour la particule entrante et sortante; w, w' sont les spineurs de Dirac correspondants, normalisés ici à l'unité; $\boldsymbol{\epsilon}$ est le quadrivecteur polarisation du photon libre. $\mathbf{M}$ groupe l'ensemble des facteurs correspondant à la partie non précisée du diagramme A ou B; c'est la même quantité dans les deux cas, à cette seule différence près que (du fait de la conservation de l'impulsion-énergie à chaque vertex) $\mathbf{M}$ est fonction de l'impulsion-énergie du photon virtuel dans le cas A, et du photon réel dans le cas B.

Soit θ l'angle de diffusion de la particule. En remplaçant d^3p' par $2\pi\sin\theta\,p'^2\,\mathrm{d}p'$, en négligeant les termes d'ordre supérieur en m/E et m/E', et enfin en posant: $\varphi = (2E'/k)\sin\theta/2$, on obtient:

$$\frac{\mathrm{d}^3p'}{(\mathbf{p}-\mathbf{p}')^4} \simeq 2\pi\,\frac{\mathrm{d}E'\sin\theta\,\mathrm{d}\theta}{E'^2(4\sin^2\theta/2+(m^2k^2/E^2E'^2))^2} = 2\pi\frac{E^2\,\mathrm{d}k}{E^2k^2}\,\frac{\varphi\,\mathrm{d}\varphi}{(\varphi^2+(m^2/E^2))^2}\,. \tag{7}$$

On conclut aisément de (7) que les valeurs de φ qui joueront un rôle prédominant dans le spectre seront telles que: $m/E \leqslant \varphi \ll 1$. D'autre part, on remarque également que l'on aura à considérer essentiellement des valeurs faibles pour l'angle θ. Dans ces conditions, on voit que φ peut être sensiblement assimilé à l'angle d'émission du photon virtuel par rapport à la direction de la particule incidente.

Notons par ailleurs que si nous appelons $\boldsymbol{k}_v$ $(=\boldsymbol{p}-\boldsymbol{p}')$ le vecteur impulsion du photon virtuel, on a:

$$\frac{|\,\boldsymbol{k}_v\,|^2-k^2}{k^2} \simeq \frac{E}{E'}\left(\varphi^2+\frac{m^2}{E^2}\right). \tag{8}$$

Le second membre est en général négligeable, donc $|\,\boldsymbol{k}_v\,| \simeq k$. Autrement dit, le photon virtuel peut être assimilé au photon réel du point de vue de l'impulsion-énergie.

Considérons maintenant le courant de transition $\mathbf{j} = \overline{w}'\boldsymbol{\gamma}w$. Nous adopterons un système d'axes orthogonaux $Oxyz$ où Ox est défini par la direction du photon virtuel et Oy est perpendiculaire au plan de diffusion. Nous aurons une expression approchée de $\mathbf{j}$ en négligeant les termes en m/E, m/E' et d'autre part

en faisant un développement en série de puissances de φ dont on ne gardera que les termes d'ordre 1. On obtient dans ces conditions:

a) Cas sans retournement de spin:

$$(9) \qquad j_x = 1 \; ; \qquad j_y = 0 \; ; \qquad j_z = \frac{2E - k}{2E'} \varphi \; ; \qquad j_0 = 1 \; .$$

b) Cas avec retournement de spin:

$$(10) \qquad j_x = \frac{k}{2E'} \varphi \; ; \qquad j_y = i \frac{k}{2E'} \varphi \; ; \qquad j_z = 0 \; ; \qquad j_0 = \frac{k}{2E'} \varphi \; .$$

Nous poserons dans le cas *a*): $\mathbf{j} = ((2E - k)/2E')\varphi \boldsymbol{\epsilon}^{(a)}$; dans le cas *b*): $\mathbf{j} = i(k/2E')\varphi \boldsymbol{\epsilon}^{(b)}$. En pratiquant sur $\boldsymbol{\epsilon}^{(a)}$ et $\boldsymbol{\epsilon}^{(b)}$ la transformation de jauge (voir [5]): $\boldsymbol{\epsilon} \to \boldsymbol{\epsilon} - (\varepsilon_0/k)\mathbf{k}$, on obtient:

$$(11) \qquad \boldsymbol{\epsilon}^{(a)} = (0, 0, 1, 0) \; ; \qquad \boldsymbol{\epsilon}^{(b)} = (0, 1, 0, 0) \; .$$

Dans les deux cas, on a donc, dans notre approximation, une polarisation transversale (soit parallèle au plan de diffusion, soit perpendiculaire à ce plan) pour le photon virtuel. On est donc en droit de considérer celui-ci comme un photon « presque réel » et de l'assimiler au photon libre.

En sommant les cas *a*) et *b*), et en remplaçant $\boldsymbol{\epsilon}^{(a)}$ et $\boldsymbol{\epsilon}^{(b)}$ par $\boldsymbol{\epsilon}$ (identifié avec le quadrivecteur polarisation du photon libre), on a:

$$(12) \qquad |\mathbf{j} \cdot \mathbf{M}|^2 = \frac{(2E - k)^2 + k^2}{4E'^2} |\boldsymbol{\epsilon} \cdot \mathbf{M}|^2 \; .$$

D'où finalement:

$$(13) \qquad \sigma_A = \frac{\alpha}{\pi} \int_0^{\varphi_{\max}} \frac{\varphi^3 \, \mathrm{d}\varphi}{(\varphi^2 + m^2/E^2)^2} \int_{k_{\min}}^{k_{\max}} \frac{\mathrm{d}k}{k^2} \frac{(2E - k)^2 + k^2}{4E^2} |\boldsymbol{\epsilon} \cdot \mathbf{M}|^2 =$$

$$= \frac{2\alpha}{\pi} \left(\ln \frac{E}{m} \varphi_{\max} - \frac{1}{2} \right) \int_{k_{\min}}^{k_{\max}} \frac{\mathrm{d}k}{k} \left(1 - \frac{k}{E} + \frac{k^2}{2E^2} \right) \sigma_B \; .$$

On a donc bien abouti à une formule du type (2), où le spectre équivalent de photons est défini par:

$$(14) \qquad P(k) \, \mathrm{d}k = \frac{2\alpha}{\pi} \left(\ln \frac{E}{m} \varphi_{\max} - \frac{1}{2} \right) \frac{\mathrm{d}k}{k} \left(1 - \frac{k}{E} + \frac{k^2}{2E^2} \right) .$$

([5]) S. S. SCHWEBER, H. A. BETHE et F. DE HOFFMANN: *Mesons and Fields*, vol. **1** (Evanston, 1955), p. 184 et suiv.

Ce résultat appelle une série de remarques:

1) Comparée à la formule semi-classique (3) ou (4), la formule (14) a évidemment un caractère plus général. Dans le cas où $k \ll E$ (cas traité précédemment dans [4]), ces formules deviennent identiques, sauf apparemment en ce qui concerne le facteur logarithmique. Mais en fait, il suffit de considérer qu'il est logique de définir le paramètre d'impact classique b comme $(k_{\text{lat}})^{-1}$, où k_{lat} est l'impulsion cédée latéralement par la particule. Donc:

$$b = (k_{\text{lat}})^{-1} \simeq (k\varphi)^{-1} ; \qquad b_{\min} \simeq (k\varphi_{\max})^{-1} , \tag{15}$$

et l'identité entre les formules (3) et (14) pour le cas $k \ll E$ devient pratiquement complète.

2) Un avantage important de notre méthode, par rapport à la méthode semi-classique, et dans une certaine mesure aussi par rapport à d'autres démonstrations plus modernes aboutissant au même type de formule (par ex., [6,7]), est de mettre en évidence un spectre angulaire, soit pour l'angle d'émission du photon virtuel, soit pout l'angle de diffusion de la particule. Ceci permet de définire une valeur moyenne (sur le spectre) de l'angle de diffusion:

$$\bar{\theta} \simeq \frac{k}{E} \ln^{-1} \frac{E}{m} , \tag{16}$$

pour une valeur donnée de k.

3) Il convient de choisir avec soin les limites $\varphi_{\max}$, $k_{\min}$, $k_{\max}$ qui dépendent chaque fois du processus envisagé. $\varphi_{\max}$ ne doit en aucun cas être supérieur à 1. $k_{\min}$ est donné par le seuil expérimental de l'effet considéré. Quant à $k_{\max}$, il faudrait en principe (conformement à notre hypothèse initiale $E, E' \gg m$) le prendre tel que $E - k_{\max} \gg m$. Toutefois, l'intégrale sur k est dans la plupart des cas peu sensible à sa limite supérieure. Pour les interactions nucléaires des électrons et des mésons μ, on peut en général prendre $\varphi_{\max} \simeq 1$, $k_{\max} \simeq E$.

4) L'approximation essentielle que nous faisons consiste à nous limiter aux petites valeurs de φ, et à ne conserver que les termes de l'ordre le plus bas lorsque nous développons le spectre en série de puissances de φ. De cette manière, nous éliminons les photons « très virtuels » (non assimilables aux photons réels): à savoir les photons « eloignés de la couche d'énergie » ($|\boldsymbol{k}_v| \gg k$) et les photons à polarisation longitudinale. On vérifie aisément que la somme des termes que nous avons négligés dans le spectre est de l'ordre de $\ln^{-1} E/m$

(6) R. B. Curtis: *Phys. Rev.*, **104**, 211 (1957).
(7) R. H. Dalitz et D. R. Yennie: *Phys. Rev.*, **105**, 1598 (1957).

par rapport au terme conservé. On peut en conclure que l'erreur relative sur la section efficace σ_A est également en général de cet ordre (en supposant σ_B connu exactement). L'erreur est évidemment d'autant plus faible que la particule est plus relativiste. En tout cas, la méthode est presque toujours satisfaisante lorsque l'on se contente d'un ordre de grandeur (notamment, lorsqu'il s'agit de vérifier théoriquement des résultats concernant les interactions de particules du rayonnement cosmique).

Notons que la méthode peut perdre sa validité dans certains cas particuliers (mis à part le fait qu'elle est inapplicable pour tout processus élastique, faute d'un processus équivalent impliquant des photons libres). Ces cas particuliers sont les suivants: *a*) On sélectionne les grands angles de diffusion. *b*) On considère un processus pour lequel les photons « très virtuels » possèdent une efficacité très supérieure à celle des photons « presque réels » qui forment la majeure partie du spectre.

Ces réserves faites, il faut souligner que notre procédé ne constitue rien d'autre qu'une approximation de petits angles, et non une approximation dipôlaire comme certains auteurs l'ont affirmé erronément ([8]).

3. – La méthode des processus quasi reels.

On peut considérer que la formule de Williams et Weizsäcker donne la probabilité d'un certain processus virtuel élémentaire traité comme un processus « quasi réel »: à savoir l'émission d'un photon virtuel par une particule chargée relativiste.

Il est possible de généraliser ce concept de processus quasi réel, et de l'étendre à tous les processus virtuels élémentaires comportant un photon et deux états de fermion (relativiste).

3·1. *Émission d'un photon par un fermion relativiste.* – Considérons le diagramme élémentaire représenté par la Fig. 2. Nous supposons qu'il est rattaché à un certain diagramme de Feynman complet; pour plus de simplicité nous admettons que le branchement se fait par une seule des trois lignes de la Fig. 2. A ces trois possibilités de branchement correspondent les trois cas:

1) Le photon est virtuel; le fermion entrant et le fermion sortant sont réels.
2) Le fermion sortant seul est virtuel.
3) Le fermion entrant seul est virtuel.

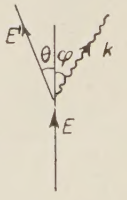

Fig. 2.

([8]) G. N. Fowler et A. W. Wolfendale: *Progress in Elementary Particle and Cosmic Ray Physics*, vol. 4 (Amsterdam, 1958), p. 170 et suiv.

Nous allons montrer que dans les trois cas (à la condition d'avoir chaque fois $E, E' \gg m$), la Fig. 2 représente un processus quasi réel, auquel nous pouvons attribuer la probabilité:

$$P(E, k)\,\mathrm{d}k = \frac{2\alpha}{\pi}\left(\ln\frac{E}{m}\varphi_{\max} - \frac{1}{2}\right)\frac{\mathrm{d}k}{k}\left(1 - \frac{k}{E} + \frac{k^2}{2E^2}\right). \tag{17}$$

La précision de cette formule est, bien entendu, d'autant meilleure que la particule est plus relativiste.

Si nous appelons A le diagramme de Feynman complet considéré, B le même diagramme amputé de la Fig. 2, les sections efficaces correspondantes σ_A et σ_B seront reliées par l'une ou l'autre des deux formules:

$$\sigma_A = \int P(E, k)\,\sigma_B\,\mathrm{d}k\,, \tag{18}$$

$$\sigma_A = \int P(E, k)\,\frac{\mathrm{d}\sigma_B}{\mathrm{d}E}\,\mathrm{d}E\,\mathrm{d}k\,, \tag{19}$$

suivant que E est une quantité fixe ou variable.

Etudions successivement les trois cas:

1) Le photon est virtuel. C'est le cas que nous avons traité dans la Section 2. La formule (17) est identique à la formule de Williams et Weizsäcker généralisée (14).

2) Le fermion sortant est virtuel. Nous avons à comparer les sections efficaces correspondant aux deux diagrammes A et B de la Fig. 3 L'application des régles de Feynman nous donne:

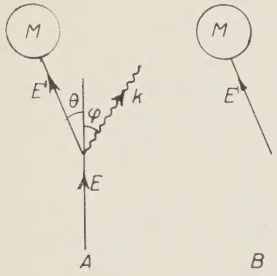

Fig. 3.

$$\sigma_A = \frac{\alpha}{4\pi^2}\int \frac{\mathrm{d}^3k}{k(\mathbf{p}^2 - \mathbf{k}^2 - m^2)^2}\,|\,M(\mathbf{p}'\cdot\boldsymbol{\gamma} + m)(\boldsymbol{\epsilon}\cdot\boldsymbol{\gamma})w\,|^2\,, \tag{20}$$

$$\sigma_B = |\,Mw'\,|^2\,. \tag{21}$$

Les notations employées ci-dessus sont les mêmes que dans la Section 2, à ceci près que la quantité M (correspondant à la partie non précisée de A ou B) est ici un spineur adjoint.

En effectuant le calcul de σ_A, nous avons:

$$\frac{\mathrm{d}^3k}{k(\mathbf{p}^2 - \mathbf{k}^2 - m^2)^2} \simeq 2\pi\,\frac{\mathrm{d}k}{kE^2}\,\frac{\sin\varphi\,\mathrm{d}\varphi}{(4\sin^2\varphi/2 + m^2/E^2)^2} \simeq 2\pi\,\frac{\mathrm{d}k}{kE^2}\,\frac{\varphi\,\mathrm{d}\varphi}{(\varphi^2 + m^2/E^2)^2}\,. \tag{22}$$

Ici encore, comme dans le cas du photon virtuel, on a à considérer des valeurs de l'angle d'émission φ qui sont essentiellement telles que $m/E \lesssim \varphi \ll 1$.

Si nous appelons $\boldsymbol{p}_v'$ $(= \boldsymbol{p} - \boldsymbol{k})$ le vecteur impulsion du fermion sortant virtuel, nous avons:

$$\frac{|\boldsymbol{p}_v'|^2 - E'^2}{E'^2} \simeq \frac{kE}{E'^2}\left(\varphi^2 - \frac{E'm^2}{kE^2}\right), \tag{23}$$

où le second membre est généralement négligeable. Donc $|\boldsymbol{p}_v'| \simeq E'$, et le fermion virtuel du diagramme A peut être assimilé au fermion réel de B du point de vue de l'énergie-impulsion.

Si maintenant nous considérons le facteur $(\mathbf{p}\cdot\boldsymbol{\gamma}+m)$ $(\boldsymbol{\epsilon}\cdot\boldsymbol{\gamma})w$ qui figure dans l'élément de matrice de A, nous pouvons obtenir une expression approchée de cette quantité en arrêtant au premier ordre son développement en série de puissances de φ, et en négligeant les termes en m/E, m/E'. On obtient ainsi, suivant les deux possibilités de polarisation du photon:

a) $\boldsymbol{\epsilon}$ parallèle au plan de diffusion,

$$(\mathbf{p}'\cdot\boldsymbol{\gamma} + m)(\boldsymbol{\epsilon}\cdot\boldsymbol{\gamma})w_s \simeq (2E - k)\varphi w_s'. \tag{24}$$

b) $\boldsymbol{\epsilon}$ perpendiculaire au plan de diffusion,

$$(\mathbf{p}'\cdot\boldsymbol{\gamma} + m)(\boldsymbol{\epsilon}\cdot\boldsymbol{\gamma})w_s \simeq ik\varphi w_{s'}'. \tag{25}$$

Nous utilisons ci-dessus les indices s, s' pour désigner les deux orientations possibles du spin du fermion. On remarque que, comme dans le cas du photon virtuel, la polarisation parallèle au plan de diffusion correspond au non-retournement du spin du fermion, la polarisation perpendiculaire au retournement du spin.

De toute façon, le fermion virtuel se représente (dans notre approximation) par le même spineur de Dirac que le fermion libre du diagramme B, auquel il peut donc être complètement assimilé.

Dans ces conditions, on trouve, en sommant le carré de l'élément de matrice de A sur les états de polarisation du photon:

$$\sigma_A = \frac{\alpha}{2\pi}\int_0^{\varphi_{\max}} \frac{\varphi^3\,\mathrm{d}\varphi}{(\varphi^2 + m^2/E^2)^2}\int_{k_{\min}}^{k_{\max}} \frac{\mathrm{d}k}{k}\,\frac{(2E-k)^2 + k^2}{E^2}\,|Mw'|^2 = \tag{26}$$

$$= \frac{2\alpha}{\pi}\left(\ln\frac{E}{m}\varphi_{\max} - \frac{1}{2}\right)\int_{k_{\min}}^{k_{\max}}\frac{\mathrm{d}k}{k}\left(1 - \frac{k}{E} + \frac{k^2}{2E^2}\right)\sigma_B,$$

en conformité avec les formules (17), (18).

3) Le fermion entrant est virtuel. Nous comparons les sections efficaces des processus A et B représentés sur la Fig 4:

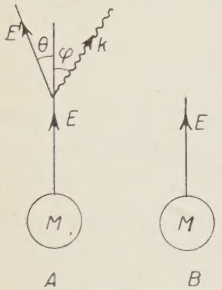

Fig. 4.

$$(27)\quad \sigma_A = \frac{\alpha}{(2\pi)^3}\int \frac{\mathrm{d}^3p'\,\mathrm{d}^3k}{k(\mathbf{p}'^2+\mathbf{k}^2-m^2)^2}\,|\,\overline{w}'(\boldsymbol{\epsilon}\cdot\boldsymbol{\gamma})(\mathbf{p}\cdot\boldsymbol{\gamma}+m)M\,|^2\,,$$

$$(28)\quad \sigma_B = \frac{1}{(2\pi)^3}\int \mathrm{d}^3p\,|\,\overline{w}M\,|^2\,.$$

Les notations employées sont les mêmes que précédemment, la quantité M (partie non précisée de A ou B) étant ici un spineur.

On obtient, avec les approximations déjà utilisées dans les deux autres cas (à savoir, essentiellement: φ petit).

$$(29)\quad \frac{\mathrm{d}^3k}{k(\mathbf{p}'^2+\mathbf{k}^2-m^2)^2} \simeq 2\pi\,\frac{\mathrm{d}k}{k}\,\frac{E^2}{E'^4}\,\frac{\varphi\,\mathrm{d}\varphi}{(\varphi^2+m^2/E^2)^2}\,,$$

et d'autre part:

$$(30)\quad w'_s(\boldsymbol{\epsilon}\cdot\boldsymbol{\gamma})(\mathbf{p}\cdot\boldsymbol{\gamma}+m) \simeq \begin{cases} \dfrac{E}{E'}\,(2E-k)\varphi\overline{w}_s & \text{pour } \boldsymbol{\epsilon} \parallel \text{ au plan} \\ i\,\dfrac{E}{E'}\,k\varphi\overline{w}_{s'} & \text{pour } \boldsymbol{\epsilon} \perp \text{ au plan}\,. \end{cases}$$

Ici encore, le fermion virtuel de A est complètement assimilable au fermion réel de B, à la fois du point de vue de son énergie-impulsion et de sa représentation.

On a finalement (en remplaçant dans σ_A les variables d'intégrations $\boldsymbol{k}$, $\boldsymbol{p}'$, par $\boldsymbol{k}$, $\boldsymbol{p}$):

$$(31)\quad \sigma_A = \frac{\alpha}{(2\pi)^4}\int \mathrm{d}^3p\,|\,\overline{w}M\,|^2\int \frac{\varphi^3\,\mathrm{d}\varphi}{(\varphi^2+m^2/E^2)^2}\int \frac{\mathrm{d}k}{k}\,\frac{(2E-k)^2+k^2}{E^2}\,;$$

d'où l'on obtient l'une ou l'autre des formules ci-dessous, suivant que E est constante ou variable dans le processus B considéré:

$$(32)\quad \sigma_A = \sigma_B\cdot\frac{2\alpha}{\pi}\left(\ln\frac{E}{m}\,\varphi_{\max}-\frac{1}{2}\right)\int_{k_{\min}}^{k_{\max}}\frac{\mathrm{d}k}{k}\left(1-\frac{k}{E}+\frac{k^2}{2E^2}\right),$$

$$(33)\quad \sigma_A = \frac{2\alpha}{\pi}\int \frac{\mathrm{d}\sigma_B}{\mathrm{d}E}\,\mathrm{d}E\left(\ln\frac{E}{m}\,\varphi_{\max}-\frac{1}{2}\right)\int_{k_{\min}}^{k_{\max}}\frac{\mathrm{d}k}{k}\left(1-\frac{k}{E}+\frac{k^2}{2E^2}\right).$$

Le résultat est conforme à (17), (18), (19).

3·2. *Création d'une paire de fermions relativistes par un photon.* – Par symétrie par rapport à l'axe des temps, on peut immédiatement passer du cas de l'émission d'un photon, que nous venons de traiter, à celui de l'absorption d'un photon par un fermion relativiste (Fig. 5). Dans ce cas, la formule (17) définit la probabilité pour qu'un fermion d'énergie finale E donnée ait subi une diffusion en absorbant un photon d'énergie k (toujours sous la condition $E, E' \gg m$).

Fig. 5.

Si l'on considère comme fixe l'énergie du photon absorbé, et non l'énergie initiale ou finale de la particule, on montre facilement que la probabilité du processus élémentaire de la Fig. 5 s'obtient alors en multipliant simplement le second membre de (17) par le facteur $E^2\,\mathrm{d}E\,(k^2\,\mathrm{d}k)^{-1}$. On a ainsi:

$$P(k, E)\,\mathrm{d}E = \frac{\alpha}{\pi}\left(\ln\frac{E}{m}\varphi_{\max} - \frac{1}{2}\right)\frac{[E^2 + (E-k)^2]\,\mathrm{d}E}{k^3}. \tag{34}$$

Dès lors, on passe au cas de la production d'une paire fermion-antifermion par un photon d'énergie k donnée, en traitant l'antifermion créé comme un fermion d'énergie négative absorbé. On a aussitôt la formule:

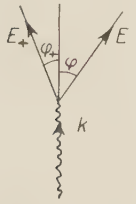

Fig. 6.

$$P(k, E)\,\mathrm{d}E = \frac{\alpha}{\pi}\left(\ln\frac{E}{m}\varphi_{\max} - \frac{1}{2}\right)\frac{[E^2 + (k-E)^2]\,\mathrm{d}E}{k^3}, \tag{35}$$

qui donne la probabilité du processus élémentaire quasi réel représenté par la Fig. 6, à condition que la paire créée soit relativiste ($E, E_+ \gg m$). Cette formule est, bien entendu, pareillement valable pour les trois possibilités de branchement de la Fig. 6, autrement dit pour les trois cas: photon virtuel, fermion virtuel, antifermion virtuel.

4. – Applications.

Nous ne nous étendrons pas ici sur les applications, maintenant nombreuses et bien connues, de la méthode de Williams et Weizsäcker proprement dite. Citons simplement parmi les plus importantes:

a) Le calcul des sections efficaces relatives aux étoiles et gerbes pénétrantes dues aux interactions nucléaires des mésons μ très énergétiques du rayonnement cosmique sous terre (voir notamment [9,10,11]).

(9) E. P. George et J. Evans: *Proc. Phys. Soc.*, A **63**, 1248 (1950); A **64**, 193 (1951); A **68**, 829 (1955).

(10) D. Kessler et R. Maze: *Nuovo Cimento*, **5**, 1540 (1957).

(11) I. B. McDiarmid: *Phys. Rev.*, **109**, 1792 (1958).

b) La comparaison entre l'électroproduction et la photoproduction de neutrons, dans une expérience faite à Stanford par Brown et Wilson ([12]).

c) La comparaison entre l'électroproduction et la photoproduction de mésons μ, dans les expériences réalisées également à Stanford par Panofsky et ses collaborateurs ([13]).

Nous allons donner ci-après quelques exemples d'application de la méthode des processus quasi réels dans des cas autres que celui du spectre de photons virtuels.

4·1. *Rayonnement de freinage d'un électron relativiste pour un angle de diffusion donné.* – Considérons une expérience où l'on veut mesurer la diffusion d'un faisceau d'électrons relativistes d'énergie E par des noyaux quelconques sous un angle Θ donné. On aura pour l'énergie E des électrons sortants un spectre (Fig. 7) qui comportera:

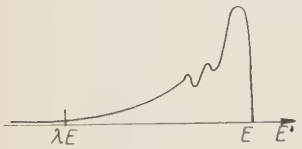

Fig. 7.

a) le pic de diffusion élastique;

b) éventuellement divers pics inélastiques;

c) le spectre de rayonnement de freinage (« bremsstrahlung ») s'étendant depuis le pic élastique jusqu'à $E' - m$ (où m est la masse de l'électron).

Nous supposons que l'on fait une coupure sur ce spectre d'énergie pour une valeur $E'_{\min} = \lambda E$, où λ est une fraction de l'unité. On admettra que Θ et λ sont choisis tels que m/E, $m/\lambda E \ll \Theta$.

Dans ces conditions, nous pouvons appliquer la méthode des processus quasi réels au calcul du rayonnement de freinage, en reliant celui-ci à la diffusion élastique, comme le montrent les deux diagrammes (*a*) et (*b*) de la Fig. 8. En effet, sur ces deux diagrammes, nous avons chaque fois au point P un processus quasi réel du type représenté par la Fig. 2.

En P, l'électron est essentiellement peu dévié, de sorte que nous pouvons sensiblement confondre l'angle total de diffusion Θ

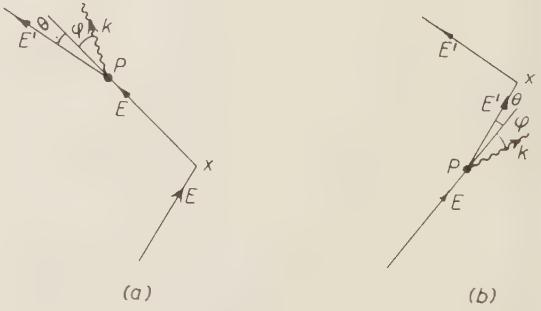

Fig. 8.

([12]) K. L. Brown et R. Wilson: *Phys. Rev.*, **93**, 443 (1954).

([13]) W. K. H. Panofsky, C. M. Newton et G. B. Yodh: *Phys. Rev.*, **98**, 751 (1955); W. K. H. Panofsky, W. M. Woodward et G. B. Yodh: *Phys. Rev.*, **102**, 1392 (1956); G. B. Yodh et W. K. H. Panofsky: *Phys. Rev.*, **105**, 731 (1957).

avec l'angle de la diffusion élastique. D'autre part, du fait que φ est essentiellement faible, on peut négliger les termes d'interférence entre les deux diagrammes: en effet, dans le cas (a) le photon est émis à peu près dans la direction de l'électron final, dans le cas (b) à peu près dans la direction de l'électron initial; les états finaux sont donc différents.

Soit σ_{tr} et σ_{el} les sections efficaces différentielles respectives du rayonnement de freinage et de la diffusion élastique pour l'angle Θ donné. On peut écrire:

$$\sigma_{tr}(E) = \sigma_{br}^{(a)} + \sigma_{br}^{(b)}, \tag{36}$$

$$\sigma_{tr}^{(a)} = \sigma_{el}(E) \cdot \int P(E, k)\, dk\,; \qquad \sigma_{tr}^{(b)} = \int P(E, k)\, dk \cdot \sigma_{el}(E')\,, \tag{37}$$

où $P(E, k)$ est donné chaque fois par la formule (17).

Comme on a $\sigma_{el}(E) \propto E^{-2}$ d'après la formule de Mott (en admettant que l'on puisse négliger l'effet de structure des noyaux), il vient:

$$\sigma_{br}(E) = \sigma_{el}(E) \cdot \int P(E, k)\, dk \left(1 + \frac{E^2}{E'^2}\right), \tag{38}$$

et en explicitant:

$$\frac{\sigma_{tr}(E)}{\sigma_{el}(E)} \quad \frac{2\alpha}{\pi}\left(\ln\frac{E}{m}\varphi_{max} - \frac{1}{2}\right)\int_{k_{min}}^{(1-\lambda)E} \frac{dk}{k}\,\frac{[E^2 + (E-k)^2]^2}{2E^2(E-k)^2}\,. \tag{39}$$

Il convient de prendre $\varphi_{max} \approx \Theta$; d'autre part, k_{min} peut être choisi aussi voisin de zéro que l'on veut. Dans ces conditions, on obtient après intégration (en negligeant les termes en k_{min}/E):

$$\frac{\sigma_{br}(E)}{\sigma_{el}(E)} = \frac{4\alpha}{\pi}\left(\ln\frac{E\Theta}{m} - \frac{1}{2}\right)\left[\ln\frac{E}{k_{min}} + f(\lambda)\right], \tag{40}$$

où

$$f(\lambda) = \frac{1}{4}\ln\frac{(1-\lambda)^4}{\lambda} + \frac{1}{4\lambda} - \frac{5}{8} + \frac{\lambda}{4} - \frac{\lambda^2}{8}\,. \tag{41}$$

Notons que le résultat ci-dessus aurait pu être obtenu également en faisant un calcul d'approximation (relativiste) à partir de la formule générale de Bethe et Heitler [14] pour le rayonnement de freinage. Un tel calcul a été effectué par Schiff [15], avec toutefois des conditions aux limites un peu différentes.

[14] H. A. Bethe et W. Heitler: *Proc. Roy. Soc.*, A **146**, 83 (1934).
[15] L. I. Schiff: *Phys. Rev.*, **87**, 750 (1952).

(Dans le cas traité par SCHIFF, une limite inférieure est imposée au moment de recul du noyau, et non à l'énergie finale de l'électron.)

Nous pouvons considérer que la formule (40) donne une correction à apporter à la section efficace de diffusion élastique, correction due à l'émission de photons réels d'énergie $k > k_{min}$: Soit $\delta' = \sigma_{tr}(E)/\sigma_{el}(E)$ ce terme correctif. Si on le considère isolément, on se trouve devant le problème apparent d'une divergence infra-rouge. Mais en fait, il faut additionner δ' avec le terme $-\delta$ calculé par SCHWINGER ([16]), et qui correspond à la somme de la correction radiative proprement dite (due à l'émission et à la réabsorption de photons virtuels) et de la correction associée à l'emission de photon réels de très basse énergie ($k < k_{min}$). Le résultat de Schwinger est:

$$(42) \qquad -\delta = -\frac{4\alpha}{\pi}\left[\left(\ln\frac{E\Theta}{m} - \frac{1}{2}\right)\left(\ln\frac{E}{k_{min}} - \frac{13}{12}\right) - \frac{17}{72} + \Phi(\Theta)\right],$$

où $\Phi(\Theta)$ est donné par:

$$(43) \qquad \Phi(\Theta) \sim \frac{1-\cos\Theta}{[2\cos(\Theta/2)(1+\cos\Theta/2)]^{\frac{1}{2}}}\left[\ln\frac{1}{2(1-\cos\Theta/2)} - \frac{1-\cos\Theta/2}{2} + 1\right].$$

On obtient ainsi la correction radiative totale $\Delta = \delta' - \delta$, qui inclut à la fois la correction radiative proprement dite et le rayonnement de freinage:

$$(44) \qquad \Delta = \frac{\alpha}{\pi}\left\{\left(\ln\frac{E\Theta}{m} - \frac{1}{2}\right)\left[\ln\frac{(1-\lambda)^4}{\lambda} + \frac{1}{\lambda} + \frac{11}{6} + \lambda - \frac{\lambda^2}{2}\right] - \frac{17}{18} - 4\Phi(\Theta)\right\}.$$

Δ peut être négatif ou positif, suivant que le paramètre λ a une valeur plus ou moins voisine de 1. Ainsi, pour des électrons d'une énergie de 200 MeV, diffusés sous un angle de 60°, on trouve: $\Delta = -7.6\%$ pour $\lambda = 0.9$; $\Delta = +2{,}3\%$ pour $\lambda = 0.5$.

4·2. *Recul des noyaux dû a la création de paires par des photons de haute énergie.* – Supposons réalisée une expérience où l'on bombarde des noyaux avec des photons d'énergie élevée et où l'on observe le recul des noyaux provoqué par la création de paires d'électrons. Nous supposons que ce recul est mesuré sous un certain angle γ par rapport au photon incident et au-dessus d'un seuil expérimental $q_{min} \gg m$ (où m est la masse de l'électron); ces conditions seraient analogues à celles de l'expérience de Tautfest et Panofsky ([17]), où l'on a ainsi

([16]) J. SCHWINGER: *Phys. Rev.*, **75**, 899 (1949).

([17]) G. W. TAUTFEST et W. K. H. PANOFSKY: *Phys. Rev.*, **105**, 1356 (1957).

mesuré la diffusion (élastique et radiative) d'électrons de haute énergie par des protons.

La section efficace différentielle de création de paires, dans les conditions indiquées, se calculera en additionnant les termes donnés par les deux diagrammes de la Fig. 9. Du fait que les angles φ, φ_+ sont essentiellement faibles, on peut négliger les termes d'interférence entre les deux diagrammes: en effet, les états finaux sont différents, puisque dans le cas (*a*) c'est le positron qui va à peu près vers l'avant, dans le cas (*b*) c'est l'électron.

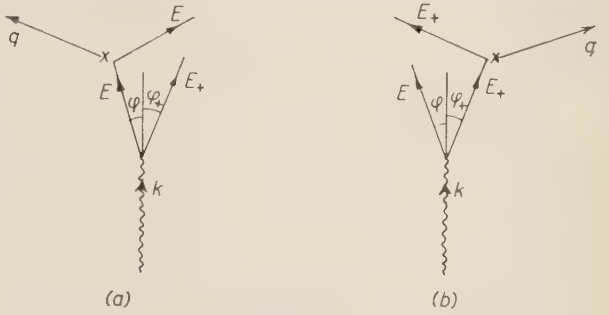

Fig. 9.

La petitesse des angles φ, φ_+ permet d'autre part de prendre: $\gamma \simeq \pi/2 - \Theta/2$, où Θ est l'angle de la diffusion élastique de l'électron ou du positron sur le noyau. Il en résulte que le recul est lié à l'énergie de l'électron ou du positron diffusé par: $q \simeq 2E\cos\gamma$ dans le cas (*a*) ou $q \simeq 2E_+\cos\gamma$ dans le cas (*b*). Il est bien entendu que l'on doit avoir *a priori*: $k > q_{\min}/2\cos\gamma$.

Du fait de la symétrie de charge, il est évident que les deux diagrammes donneront le même résultat. D'où:

$$\sigma_{\text{pair}}(k) = \sigma^{(a)} + \sigma^{(b)} = 2\sigma^{(a)} = 2\int P(k, E)\,\mathrm{d}E\cdot\sigma_{\text{el}}(E)\,, \tag{45}$$

où $P(k, E)$ est donné par la formule (35).

On a par ailleurs, d'après la formule de Mott: $\sigma_{\text{el}}(E) = \sigma_{\text{el}}(k)\cdot k^2/E^2$, où $\sigma_{\text{el}}(k)$ est la section efficace différentielle de diffusion élastique pour un électron d'énergie k, diffusé sous l'angle Θ. D'où:

$$\sigma_{\text{pair}}(k) = \sigma_{\text{el}}(k)\cdot\frac{2\alpha}{\pi}\int_{E_{\min}}^{E_{\max}}\left(\ln\frac{E}{m}\,q_{\max} - \frac{1}{2}\right)\frac{E^2 + (k-E)^2}{E^2}\,\frac{\mathrm{d}E}{k}\,. \tag{46}$$

On a: $E_{\min} = q_{\min}/2\cos\gamma$; d'autre part, on peut prendre: $E_{\max} = k$, et $q_{\max} \simeq \Theta \simeq 2\cos\gamma$; enfin, on ne commet pas d'erreur sensible en substituant $E_{\min}$ à E sous le logarithme. Dans ces conditions, l'intégration donne:

$$\sigma_{\text{pair}}(k) = \sigma_{\text{el}}(k)\cdot\frac{4\alpha}{\pi}\left(\ln\frac{q_{\min}}{m} - \frac{1}{2}\right)\left(\frac{k\cos\gamma}{q_{\min}} - \ln\frac{2k\cos\gamma}{q_{\min}} + \frac{1}{2} - \frac{q_{\min}}{2k\cos\gamma}\right). \tag{47}$$

4·3. *Rayonnement de freinage interne dans la désintégration du méson* μ. – Au cours des dernières années, divers auteurs [18-20] ont étudié le problème des corrections radiatives dans la désintégration du méson μ. D'autre part, la possibilité de détecter expérimentalement l'effet de rayonnement de freinage interne a été examinée [21].

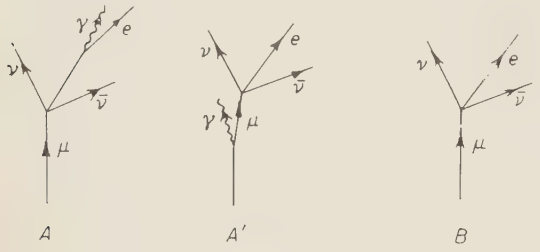

Fig. 10.

Ce dernier problème revient à évaluer le rapport de branchement entre les réactions $\mu \to e + \nu + \bar{\nu} + \gamma$ et $\mu \to e + \nu + \bar{\nu}$, en fixant un seuil expérimental raisonnable pour l'énergie du photon émis.

Il s'agit donc de comparer les diagrammes A et A' de la Fig. 10 avec le diagramme B. Nous plaçant dans le système de référence où le méson μ est au repos, nous pouvons admettre *a priori* que la contribution de A' est négligeable par rapport à celle de A.

Considérons le processus B. Soit E l'énergie de l'électron créé, m sa masse; son énergie maximum est: $W \simeq \frac{1}{2} m_\mu \simeq 100m$. Le spectre d'énergie est donné par la formule de Michel:

$$f(E)\,\mathrm{d}E = \frac{4E^2}{3W^4}[9(W-E) + 2\varrho(4E-3W)]\,\mathrm{d}E\,, \tag{48}$$

où ϱ a été défini à l'origine comme un paramètre compris entre 0 et 1, sa valeur étant fonction de la nature du couplage entre les quatre fermions; la valeur généralement adoptée à l'heure actuelle est $\varrho \sim 0.75$ (correspondant au couplage $V-A$ proposé par Feynman et Gell-Mann [22]).

Il résulte de l'expression du spectre (48) que la région où l'on a $E \gg m$ joue un rôle prépondérant. Ceci permet d'appliquer notre méthode, et de relier le diagramme A à B en traitant l'émission du photon par l'électron comme un processus élémentaire quasi réel.

Dans ces conditions, compte tenu du fait que pour le processus B le spectre total $\int f(E)\,\mathrm{d}E$ est normalisé à l'unité, le rapport de branchement $\delta = \sigma_A/\sigma_B$

[18] R. E. Behrends, R. J. Finkelstein et A. Sirlin: *Phys. Rev.*, **101**, 866 (1956).
[19] S. M. Berman: *Phys. Rev.*, **112**, 267 (1958).
[20] T. Kinoshita et A. Sirlin: *Phys. Rev.*, **113**, 1652 (1959).
[21] T. Kinoshita et A. Sirlin: *Phys. Rev. Lett.*, **2**, 177 (1959).
[22] R. P. Feynman et M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958).

sera défini par

$$\delta = \int_{E_{\min}}^{W} f(E)\,\mathrm{d}E \int_{k_{\min}}^{k_{\max}} P(E, k)\,\mathrm{d}k\,, \tag{49}$$

où $P(E, k)$ est donné par la formule (17).

On attribuera au seuil $k_{\min}$ une valeur de l'ordre de quelque masses électroniques. D'autre part, on peut prendre: $k_{\max} \simeq E$, $\varphi_{\max} \simeq 1$. D'où:

$$\int_{k_{\min}}^{k_{\max}} P(E, k)\,\mathrm{d}k \simeq \frac{2\alpha}{\pi}\left(\ln\frac{E}{m} - \frac{1}{2}\right)\left(\ln\frac{E}{k_{\min}} - \frac{3}{4}\right), \tag{50}$$

ayant négligé les termes en $k_{\min}/E$.

Posons maintenant; $x = E/W$; $l = \ln k_{\min}/m$, et notons que: $\ln W/m \simeq 4.6$. L'équation (50) prend la forme:

$$\int_{k_{\min}}^{k_{\max}} P(E, k)\,\mathrm{d}k = \frac{2\alpha}{\pi}\,(a + b\ln x + \ln^2 x)\,, \tag{51}$$

avec: $a = 16 - 4.1\,l$; $b = 8 - l$. D'où:

$$\delta = \frac{8\alpha}{3\pi}\int_{x_{\min}}^{1}[9(x^2 - x^3) + 2\varrho(4x^3 - 3x^2)]\,(a + b\ln x + \ln^2 x)\,\mathrm{d}x\,. \tag{52}$$

On a $x_{\min} \approx k_{\min}/W \ll 1$; il s'ensuit que l'on aura une bonne approximation en prenant une limite d'intégration inférieure nulle. On obtient ainsi:

$$\delta = \frac{8\alpha}{3\pi}\left[\frac{3}{4}\,a - \frac{7}{16}\,b + \frac{37}{96} + \varrho\left(\frac{b}{6} - \frac{7}{36}\right)\right] = [5.5 - 1.6\,l + \varrho\,(0.7 - 0.1\,l)]\,\%\,. \tag{53}$$

On constante que le résultat est peu sensible à la valeur de ϱ. Pour $\varrho = 0.75$, le rapport de branchement se réduit à:

$$\delta = (6 - 1.7\,l)\,\%\,. \tag{54}$$

En prenant par exemple $k_{\min} = 2\,m$, on obtient $\delta = 4.8\,\%$. Ce chiffre est en excellent accord avec le résultat de KINOSHITA et SIRLIN [21], obtenu à la suite de calculs plus complexes.

4·4. *Rayonnement de freinage interne dans la désintégration β de* ^{12}B *et* ^{12}N. – Notre méthode peut évidemment s'étendre au calcul du rayonnement de freinage interne dans la désintégration β, chaque fois que le bilan d'énergie est élevé par rapport à la masse de l'électron. Ceci est le cas, entre autres, dans les réactions ^{12}B(β^-)^{12}C et ^{12}N(β^+)^{12}C, où l'énergie maximum de l'électron négatif ou positif émis est respectivement de 13.9 et 17.1 MeV.

Il a été récemment suggéré ([23,24]) que la comparaison expérimentale entre ces deux réactions pouvait fournir un test de validité de la théorie de Feynman et Gell-Mann ([22]) sur l'intéraction vectorielle dans la désintégration β. Or, le calcul comparé des deux spectres fait intervenir différentes corrections électromagnétiques (voir ([25])) dont l'une est due au rayonnement de freinage interne. En effet, le taux de branchement de ce processus (par rapport à la désintégration β non radiative) n'est pas le même pour ^{12}B et ^{12}N, en raison de la différence entre les bilans d'énergie.

Supposons que l'on mesure l'un et l'autre spectre pour une énergie déterminée E ($\gg m$) de l'électron. Dans ce cas, le taux de branchement du rayonnement de freinage interne (soit pour ^{12}B, soit pour ^{12}N) est défini par:

$$\delta = [f(E)]^{-1} \cdot \int_{k_{\min}}^{k_{\max}} f(E+k)\, P(E+k,\, k)\, \mathrm{d}k\,, \tag{55}$$

où le spectre $f(E)$ est donné en première approximation par la formule de Fermi ([26]) pour les transitions β permises:

$$f(E)\,\mathrm{d}E \propto E^2(W-E)^2\,\mathrm{d}E\,, \tag{56}$$

W étant l'énergie maximum de l'électron. D'autre part, d'après notre formule (17), $P(E+k, k)$ est défini par:

$$P(E+k,\, k)\,\mathrm{d}k = \frac{\alpha}{\pi}\left(\ln\frac{E+k}{m}\,\varphi_{\max} - \frac{1}{2}\right)\frac{\mathrm{d}k}{k}\,\frac{2E^2+2Ek+k^2}{(E+k)^2}\,. \tag{57}$$

On peut valablement substituer E à $E+k$ sous le logarithme, et prendre $\varphi_{\max} \simeq 1$. Notons d'autre part que $k_{\max} = W - E$. Il vient dans ces conditions:

$$\delta = E^{-2}k_{\max}^{-2}\cdot\frac{\alpha}{\pi}\left(\ln\frac{E}{m} - \frac{1}{2}\right)\int_{k_{\min}}^{k_{\max}}\frac{\mathrm{d}k}{k}\,(k_{\max}-k)^2(2E^2+2Ek+k^2)\,. \tag{58}$$

([23]) M. Gell-Mann: *Phys. Rev.*, **111**, 362 (1958).
([24]) M. Morita: *Phys. Rev.*, **113**, 1584 (1959).
([25]) M. Gell-Mann et S. M. Berman: *Phys. Rev. Lett.*, **3**, 99 (1959).
([26]) E. Fermi: *Zeits. Phys.*, **88**, 161 (1934).

En négligeant les termes en $k_{\min}/E$ et $k_{\min}/k_{\max}$, l'intégration donne:

$$\delta = \frac{2\alpha}{\pi}\left(\ln\frac{E}{m} - \frac{1}{2}\right)\left(\ln\frac{k_{\max}}{k_{\min}} - \frac{3}{2} + \frac{k_{\max}}{3E} + \frac{k_{\max}^2}{24E^2}\right). \tag{59}$$

Il s'ensuit que le terme correctif qui intervient dans la comparaison qui nous intéresse est de la forme:

$$\delta_{\mathrm{B}} - \delta_{\mathrm{B}} = \frac{2\alpha}{\pi}\left(\ln\frac{E}{m} - \frac{1}{2}\right)\left(\ln\frac{W_{\mathrm{B}} - E}{W_{\mathrm{N}} - E} + \frac{W_{\mathrm{B}} - W_{\mathrm{N}}}{4E} + \frac{W_{\mathrm{B}}^2 - W_{\mathrm{N}}^2}{24E^2}\right), \tag{60}$$

où nous avons utilisé les indices B et N pour les quantités relatives à la désintégration de ^{12}B et ^{12}N respectivement.

Ce résultat ne diffère pas sensiblement de celui donné par GELL-MANN et BERMAN (formule (5) de la référence [25]); ces auteurs avaient utilisé une méthode moins générale.

5. – Conclusion.

Les exemples d'application qui précèdent mettent en lumière l'utilité de la méthode des processus quasi réels. Bien que dans tous les cas traités d'autres méthodes puissent être employées également et conduisent aux mêmes résultats, il est manifeste que la nôtre apporte une simplification importante. Elle permet chaque fois de déterminer l'ordre de grandeur associé à un processus relativiste sans en faire le calcul détaillé; on le relie directement à un processus d'ordre inférieur dont on connait le résultat, et le calcul devient alors presque immédiat en utilisant nos formules.

Il est certainement souhaitable d'étendre encore le domaine d'application de ce type de méthode; il nous semble que des généralisations pourraient être tentées dans diverses directions.

Ainsi, on peut envisager l'extension de notre procédé aux effets qui impliquent les interactions pion-nucléon. Si l'on admettait que l'on peut ici encore appliquer la théorie du couplage faible et représenter valablement un processus donné par le diagramme correspondant à l'ordre le plus bas, le problème ne serait pas plus compliqué qu'en électrodynamique [27]. On calculerait alors

[27] Le fait que le pion (contrairement au photon) possède une masse ne constitue pas une difficulté sérieuse, du moment que cette masse est faible par rapport à celle du nucléon (voir G. MORPURGO: *Nuovo Cimento*, **6**, 504 (1949)).

aisément la probabilité associée au processus quasi réel représenté par un diagramme élémentaire comportant deux lignes de fermion (relativiste) et une ligne de boson; on obtiendrait ainsi une formule simple qui serait analogue à la formule de Williams et Weizsäcker pour le cas fermion-photon. Ceci permettrait de rattacher par exemple un processus dû à l'interaction mésique d'un nucléon relativiste au processus équivalent produit par des mésons π libres. Des tentatives semblables ont d'ailleurs déjà été faites par divers auteurs ([28,29]).

La difficulté vient évidemment ici du couplage fort dont la présence nous interdit de négliger les diagrammes d'ordre supérieur. Il semble que l'inclusion obligatoire des corrections radiatives mésiques complique considérablement le problème, en faisant apparaître, en dehors du terme «quasi réel» (correspondant au pôle du propagateur de la particule virtuelle) d'autres termes non négligeables ([30]).

Un autre essai de généralisation qui pourrait être tenté consisterait à chercher dans quelle mesure on peut étendre éventuellement le principe de notre méthode aux processus non-relativistes. Dans cette perspective, il pourrait être commode d'utiliser des transformations de Lorentz ayant pour effet de passer chaque fois du système du laboratoire à un système de référence dans lequel toutes les particules en présence seraient extrême-relativistes.

Il nous paraît certain, de toute manière, que le concept des processus quasi réels pourra être appliqué utilement dans de nombreux cas où l'on cherchera à calculer de façon simple les ordres de grandeur d'effets complexes étudiés en physique expérimentale moderne.

* * *

Je désire signaler que la partie de ce travail qui est exposée à la Sect. 2 a été réalisée en collaboration avec le Dr. Dan Kessler, de l'Institut Weizman (Rehovot, Israël). Qu'il trouve ici l'expression de ma profonde gratitude.

Je remercie également le Professeur M. Gell-Mann et le Dr. D. Berley, actuellement au Laboratoire de Physique Atomique du Collège de France, qui m'ont fait de très utiles suggestions.

Enfin, j'exprime ma sincère reconnaissance au Centre National de la Recherche Scientifique (France), sous les auspices duquel j'ai accompli le présent travail.

([28]) G. F. Chew et F. E. Low: *Phys. Rev.*, **113**, 1640 (1959).
([29]) E. Ferrari: *Nuovo Cimento*, **13**, 1285 (1959) et **15**, 652 (1960).
([30]) M. Gell-Mann: communication privée.

RIASSUNTO (*)

Proponiamo un metodo semplificato di calcolo approssimato per i processi relativistici in elettrodinamica quantistica. Il nostro punto di partenza è una generalizzazione teorica nel campo quantistico del metodo semiclassico di Williams-Weizsäcker, che definisce uno spettro fotonico equivalente per una particella relativistica carica. Poi mostriamo che si può scrivere una formula del tipo Williams-Weizsäcker per esprimere la probabilità associata ad ogni diagramma elementare contenente due linee di fermioni ed una di fotoni, purchè entrambi gli stati fermionici siano relativistici. Si danno molti esempi di casi in cui questo concetto è applicato per calcolare processi di ordine superiore riferendoli direttamente ad effetti più semplici. I casi trattati sono: 1) bremsstrahlung di una particella relativistica ad un dato angolo; 2) rinculo nucleare per creazione di coppie da fotoni di alta energia; 3) bremsstrahlung interna nella disintegrazione del mesone μ; 4) correzioni radiative negli spettri β di raffronto di ^{12}B e ^{12}N. In conclusione si discutono brevemente ulteriori possibilità di estendere il metodo (specialmente alla teoria dei mesoni π).

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

Hypersensibilisation des émulsions nucléaires Ilford par la triéthanolamine.

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(ricevuto il 16 Aprile 1960)

Summary. — It is possible to improve by a factor of the order of 1.7 the sensitivity of K-5 emulsions using appropriate treatment based on a pH 9.5 solution of triethanolamine and citric acid.

1. – Introduction.

La triéthanolamine (T.E.A.) est reconnue et utilisée depuis de longues années comme sensibilisateur photographique. Récemment des physiciens soviétiques [1] ont montré, sur leurs propres émulsions nucléaires, qu'il était possible d'utiliser la T.E.A. comme hypersensibilisateur en l'introduisant dans l'émulsion après sa fabrication. On obtient ainsi un accroissement du nombre de grains développables le long des trajectoires des particules faiblement ionisantes, sans accroissement prohibitif du voile.

Nous avons constaté qu'il était possible d'obtenir des résultats analogues avec les émulsions nucléaires Ilford. Il nous a semblé utile d'étudier ce phénomène et de mettre au point un traitement simple, efficace et reproductible permettant de réaliser cette hypersensibilisation.

[1] K. S. Bogomolov, Rudiskaya, A. A. Sirotinskaya, A. P. Zhdanov, A. L. Kartuzanski, I. V. Ryzkova et L. I. Shur: *Žurn. Nauk Prikl. Fot. Kine*, **3**, no. 1 (1958); D. M. Samoielovich: *Žurn. Nauk Prikl. Fot. Kine*, **3**, no. 4, 284 (1958); D. Breido: *Žurn. Nauk Prikl. Fot. Kine*, **3**, no. 3, 224 (1958).

L'intérêt que peuvent présenter, dans différents problèmes, les émulsions ainsi traitées est lié essentiellement à la facilité d'observation des trajectoires d'électrons de toutes énergies et des particules relativistes.

Pour une même perte d'énergie, un nombre plus grand de grains sont développés le long de la trajectoire. Les électrons jusqu'à 1 MeV donnent des traces presque continues ce qui permet de les suivre en dépit de leurs nombreux changements de direction; les particules relativistes ont des trajectoires nettement mieux définies, ce qui permet de les suivre en présence d'un fond important et de localiser leur origine sans ambiguïté.

En outre, il est possible de ne procéder au traitement à la T.E.A. qu'au moment de l'utilisation de l'émulsion, ce qui permet, par exemple, de ne conférer à une émulsion peu sensible la sensibilité aux électrons qu'au moment où cela est nécessaire et d'éviter auparavant l'accumulation de trajectoires dues au rayonnement cosmique.

Il est également possible que l'on obtienne ainsi une discrimination meilleure entre les particules au minimum d'ionisation et celles au plateau, l'accroissement du nombre de grains pouvant être mesuré dans des conditions statistiquement plus favorables.

Par contre, il est bien évident que de telles émulsions seront moins favorables à la discrimination des particules de faible énergie fortement ionisantes, les trajectoires étant saturées pour des énergies plus faibles.

Il s'agit donc d'un traitement auquel il sera intéressant d'avoir recours pour certains problèmes mais qui peut présenter dans d'autres cas des inconvénients, de même que certaines études sont plus aisées avec des émulsions non sensibles aux particules faiblement ionisantes.

2. – Conditions d'utilisation.

a) *Généralités* – Si l'on imprègne une émulsion nucléaire Ilford d'une solution aqueuse de T.E.A. à 2.5%, on observe immédiatement un accroissement de sensibilité, aussi bien pour la lumière que pour les particules relativistes. Toutefois les résultats ne sont pas toujours reproductibles et on obtient quelquefois un voile général prohibitif.

Pour pallier ces inconvénients, les auteurs soviétiques préconisent de sécher rapidement les plaques, de les relaver après le traitement et d'utiliser un stabilisateur dont la nature n'est pas précisée. Nous avons constaté qu'effectivement la rapidité du séchage est importante et que le relavage permet de conserver les plaques traitées pendant plusieurs semaines. Sans relavage, un voile général apparaît très rapidement.

Nos essais ont porté sur la nature du stabilisateur le plus favorable et sur les variations des résultats en fonction du pH de la solution utilisée. Nous

avons utilisé essentiellement des émulsions K-5, L-4 et L-2 de différentes épaisseurs. Les irradiations ont été faites soit par la lumière, soit par des électrons de radioélement, soit par des particules relativistes obtenues au Synchrotron de Saclay, dans un canal sélectionnant des mésons π^- d'impulsion moyenne 1.1 GeV/c.

Les émulsions étaient développées suivant la méthode que nous utilisons habituellement [2]. Dans ces conditions, on obtient de 28 à 32 grains par 100 μm pour les particules relativistes dans des émulsions K-5 normales développées immédiatement après l'irradiation. L'accroissement du nombre de grains dû à l'action de la T.E.A. est de l'ordre de 15 à 20.

Lorsqu'on plonge des émulsions nucléaires dans l'eau, la glycérine qui y est normalement contenue est dissoute. Pour éviter cela, toutes les solutions utilisées contiennent 5% de glycérine.

b) Stabilisation par l'acide citrique. – Nous avons souvent constaté l'intérêt de l'acide citrique et de ses sels dans le traitement des émulsions nucléaires, aussi bien pour l'introduction de sels dans l'émulsion par imprégnation que pour le développement et le fixage. C'est pourquoi il nous a semblé naturel d'essayer l'acide citrique comme stabilisateur.

L'adjonction d'acide citrique a immédiatement amélioré les résultats. Les essais sont devenus reproductibles et le voile de fond a diminué.

Tableau I.

	K-5 600 μm		L-4 400 μm
	grains/100 μm	grains fond/1000 μm³	grains/100 μm
Témoin	29.7	1.3	15
Eau pure	20	1.2	pas de traces
Eau-acide citrique pH=6.6	29.2	1.3	13
Eau-T.E.A. pH=9.5	48	3	25
Eau-T.E.A.-acide citrique pH=9.5	49.2	2	25
Eau-T.E.A.-acide citrique pH=6.6	35.6	1	—

Le Tableau I donne les résultats dans le cas d'émulsions K-5 600 μm et L-4 400 μm pour des solutions eau-acide citrique-T.E.A.-glycérine de différentes compositions. On peut faire les remarques suivantes:

1) Les séjours dans l'eau seule entraînent une perte du nombre de grains par rapport à l'émulsion témoin.

[2] H. Faraggi, A. Garin et M. Lévi: *Nuovo Cimento*, in press.

2) Cette perte est compensée si l'on utilise une solution eau-acide citrique à pH 6.6. On retrouve alors le même nombre de grains que pour l'émulsion témoin.

3) Les meilleurs résultats sont obtenus pour une solution à pH 9.5.

4) Si l'on diminue le pH par adjonction supplémentaire d'acide citrique l'effet est notablement réduit.

5) A pH 9.5, l'adjonction d'acide citrique diminue le voile de fond et augmente très légèrement le nombre de grains. En outre, on parvient ainsi à réellement stabiliser le procédé.

Nous avons obtenu des résultats tout à fait analogues dans la sensibilité à la lumière: en particulier le séjour dans l'eau entraîne une désensibilisation qui est compensée par l'adjonction d'acide citrique et le rapport « signal/bruit » est amélioré par la stabilisation de la solution à l'aide d'acide citrique.

Pour les émulsions de 100 et 200 μm, il n'y a pas de différences entre le nombre de grains en surface et en profondeur. Pour les émulsions de 600 μm, il y a une diminution en surface qu'on observe peu dans les émulsions non traitées. Ceci est illustré dans le Tableau II. Nous n'avons pas étudié systématiquement cet effet qui semblerait indiquer une moindre résistance à la corrosion pour les grains des émulsions hypersensibilisées.

Tableau II.

	Nombre de grains/100 μm			
			K-5 600 μm	
	K-5 100 μm	K-5 200 μm	surface	milieu
Témoin	29	31.5	29.2	29.7
pH 9.5	50.5	50	38.5	49.3
pH 6.6	33	50	30	35.6
T.E.A. sans acide citrique pH 9.5	33	50	38	48

c) *Essais d'autres solutions.* – Nous avons essayé de remplacer la T.E.A par une solution alcoolique de soude. On obtient effectivement une hypersensibilisation, non reproductible, avec un accroissement considérable du voile. En outre, l'effet disparaît complètement si l'on relave les plaques avant de les utiliser. Le rôle de la T.E.A. ne peut donc être dû uniquement à son action basique et à la présence de fonction alcool.

Si l'on remplace l'acide citrique par l'acide acétique, les traces au minimum cessent d'être développables même en présence de T.E.A.

Nous avons également essayé de substituer à l'acide citrique, ou de lui ajouter, d'autres composés présentant des fonctions amine ou des fonctions alcool:

1) *Benzotriazol*: Celui-ci est connu comme sensibilisateur photographique. Il contient des fonctions amine sans fonctions alcool. Une solution de T.E.A.-benzotriazol à pH 9.5 ramène à 33 le nombre de grains par 100 μm (contre 50 pour la solution T.E.A.-acide citrique). Une solution T.E.A.-benzotriazol-acide citrique ne donne également que 35 grains. L'adjonction de benzotriazol a donc un effet destructif.

2) *Alcool*: L'adjonction d'alcool à la T.E.A. au lieu d'acide citrique ainsi que l'adjonction d'alcool et d'acide citrique ramène le nombre de grains de 48 à 32. Il y a donc également destruction de l'effet par adjonction d'alcool pour un même pH de la solution.

d) *Emulsions de type* L-2. – Les émulsions du type L-2, non sensibles aux particules faiblement ionisantes, deviennent sensibles aux électrons mous d'une source de ^{40}Ca, ^{192}Ir lorsqu'elles sont traitées par la solution T.E.A.-acide citrique à pH 9.5.

En outre, les trajectoires de protons de quelques microns deviennent beaucoup plus continues et se discernent beaucoup plus aisément du voile de fond. Cette propriété a été utilisée pour la spectroscopie des neutrons. Par contre, la discrimination alpha-protons pour des traces de faible énergie devient plus difficile.

e) *Mode opératoire.* – En résumé, la routine qui nous est apparue la plus favorable est la suivante:

1) Imprégnation pendant 40 minutes dans une solution à pH 9.5 ayant la composition suivante:

eau	100 cm³
T.E.A.	2.5 g
glycérine	5 cm³
acide citrique	0.8 cm³ d'une solution à 0.1%

2) Séchage: Celui-ci doit être effectué aussi rapidement que possible. Nous utilisons un appareil comportant une paroi froide et une paroi chaude.

3) Après 3 heures de séchage, les émulsions sont relavées dans une solution d'eau-phosphate de soude et acide citrique à pH 8.

4) Séchage: Les émulsions ainsi traitées peuvent être conservées au moins trois semaines avant leur irradiation. Nous n'avons pas fait d'essais plus prolongés.

Nous n'avons pas utilisé non plus d'émulsions de surface supérieure à 40 cm^2 ni d'épaisseur supérieure à 600 μm. Par ailleurs tous les essais ont porté sur des émulsions coulées sur verre.

3. – Discussion.

Les essais décrits ci-dessus, bien que très incomplets, fournissent un procédé commode pour accroître la sensibilité des émulsions nucléaires Ilford. Les résultats obtenus ne nous permettent sans doute pas d'élucider complètement le mécanisme de l'action de la T.E.A. sur la sensibilité des grains de bromure d'argent.

Diverses interprétations ont été proposées [1] mettant en jeu différentes théories de formation de l'image latente photographique due aux particules chargées. Indépendamment de toute théorie, l'action de la T.E.A. se traduit par le fait que les grains de l'émulsion hypersensibilisée sont rendus développables pour une moindre perte d'énergie de la particule dans la traversée du grain ou au voisinage immédiat de celui-ci.

Une partie de cette action est certainement liée au pH de la solution puisque la neutralisation par l'acide citrique fait disparaître partiellement l'effet observé. Toutefois les émulsions chargées en bore, qui sont fortement basiques, ont une action connue sur le renforcement du développement sans accroître pour autant la sensibilité; l'action de la soude est totalement éliminée par le lavage et le benzotriazol détruit l'effet. En outre, le rôle de l'acide citrique n'est pas non plus négligeable et l'alcool ne peut lui être substitué.

Il est donc probable que la T.E.A. et l'acide citrique interviennent à la fois par leurs fonctions et par leur structure en liaison avec la gélatine et le bromure d'argent. Bogomolov [3] a montré que la gélatine joue un rôle important dans l'action de la T.E.A.

Par ailleurs, il est connu que, lors de la maturation des émulsions, une partie de la gélatine peut diffuser ou être retenue à l'intérieur des grains. Il est alors permis de supposer qu'une partie de la T.E.A. puisse former des combinaisons stables avec la gélatine et diffuser à l'intérieur des grains tandis qu'une autre fraction est adsorbée à la surface. Lors du lavage, la T.E.A. superficielle serait éliminée.

La T.E.A. fixée pourrait réduire le bromure d'argent, les ions brome étant acceptés par les fonctions alcool. On obtiendrait ainsi des germes supplémentaires rendant les grains développables pour un parcours plus faible de la

[3] K. S. Bogomolov: *Colloque Intern. de Photographie Corpusculaire* (Montréal, 1958).

particule à travers le cristal et n'entraînant pas un accroissement prohibitif du voile.

Il faudrait sans doute de nombreuses expériences pour élucider le mécanisme de cet effet qui, de même que celui de la formation de l'image latente, peut donner lieu à de multiples interprétations. Quoi qu'il en soit, les ressources nouvelles offertes aux expérimentateurs par ce procédé nous ont paru suffisamment intéressantes pour justifier une étude essentiellemente empirique.

RIASSUNTO (*)

È possibile migliorare di un fattore 1.7 circa la sensibilità delle emulsioni K-5 con un appropriato trattamento usando una soluzione di trietanolamina-acido citrico di pH 9.5.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

A Method for Detecting the Possible Existence of High Energy Deuterons in the Primary Cosmic Radiation.

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(ricevuto il 19 Aprile 1960)

Summary. — A method which can be used for detecting the presence of high energy deuterons in the primary cosmic radiation is described. The suggested method takes advantage of the fact that when only the neutron of the incoming deuteron takes part in a collision and gives rise to a star in a nuclear emulsion, the proton continues in the forward direction almost undeviated. Such stars referred to as (d, p) stars, can provide evidence that the incoming particle is a deuteron. In order to use this method it is necessary to know: (i) the interaction mean free path λ_d of high energy deuterons in nuclear emulsions; and (ii) the proportion $f(d, p)$ of deuteron induced interactions in which only the neutron takes part. These have been determined to be $\lambda_d = 15.8$ cm and $f(d, p) = 0.4$. The suggested method is applicable if deuterons constitute $\geqslant 5\%$ of the singly charged particles in the primary radiation.

1. – Introduction.

About 90% of the cosmic ray particles arriving at the top of the earth's atmosphere are singly charged. The relative proportion of protons and deuterons amongst these singly charged particles, however, is not known. (The proportion of tritons is also not known, but can be considered to be negligible because of their relatively short half life of about 12 years; this statement

does not refer to particles of solar origin which arrive at times of solar flares.) It is considered generally that deuterons, as also the light elements lithium, beryllium and boron, are extremely rare in the Universe (1). If this is so, then the deuterons and the light elements observed in the cosmic radiation entering the earth's atmosphere (hereafter referred to as the primary radiation) should have been produced in collisions of heavy nuclei with atoms in interstellar and interplanetary space. Therefore, if one could experimentally determine the ratio of the number N_d of deuterons to the number N_s of singly charged particles in the primary beam, it will provide information regarding the amount of matter traversed by cosmic rays between their source and the earth; the information obtained will be similar to that derived from a determination of the ratio of the number N_L of light nuclei to the number, N_M, of medium type of nuclei (C, N, O and F).

We suggest in this paper a possible experiment to determine the quantity N_d/N_s in the relativistic energy region. We also describe some experiments we have carried out to estimate the parameters needed for a determination of N_d/N_s by this method. These results, though not very accurate, are given as an illustration of the method. The actual experiment will be conducted using a stack exposed at high altitude near the geomagnetic equator.

2. – Method.

Appa Rao *et al.* (2) have reported detailed measurements of the angular distribution of shower particles emerging from disintegrations induced in nuclear emulsions by α-particles of energy > 5 GeV/nucleon. They found that 95% of the shower particles emitted within an angle of $2 \cdot 10^{-2}$ radians were singly charged fragments which were part of the original α particle and which did not participate in the interaction. It is reasonable to assume that, in similar nuclear interactions caused by deuterons of the same energy per nucleon, the proton, if it does not take part in the collision process, would emerge within the small angle, $2 \cdot 10^{-2}$ rads., determined from the α particle interactions. This characteristic type of interaction induced by a singly charged particle, hereafter referred to as a « (d, p) *star* » provides evidence that the primary particle was a deuteron (or triton). This method can therefore be used to determine the percentage of deuterons among the singly charged particles of the primary cosmic radiation.

(1) H. E. Suess and H. C. Urey: *Rev. Mod. Phys.*, **28**, 53 (1956).

(2) M. V. K. Appa Rao, R. R. Daniel and K. A. Neelakantan: *Proc. Ind. Acad. Sci.*, **43**, 181 (1956).

In order to estimate the proportion of high energy deuterons in the primary radiation from the number of (d, p) *stars* observed in nuclear emulsion, it is necessary to know the following parameters:

a) the proportion, f(d, p), of deuteron produced stars in nuclear emulsions in which the proton does not interact;

b) the mean free path λ_d, for nuclear interactions of high energy deuterons in nuclear emulsions.

Since there are no accelerating machines at present which produce relativistic deuterons, we have estimated λ_d and f(d, p) in the following indirect manner.

Determination of λ_d and f(d, p). – QUARENI and ZORN [3] have carried out an investigation on interactions produced in nuclear emulsions by α-particles of total energy 360 MeV. In this experiment, they identified 20 singly charged fragments which emerged from the α-particle interactions with the same velocity as the incident particle; these are particles which have not taken part in the collision processes. Amongst these they found 7 protons, 12 deuterons and 1 triton. Therefore, it seemed that the interactions of α-particles would be a good source of deuterons. From similar considerations, it is reasonable to assume that heavy primary interactions would also be a good source of deuterons.

The number of tritons observed by QUARENI and ZORN (and also in this experiment - see Section 3·1) amongst the singly charged fragments is small ($\sim 5\%$); we will, therefore, consider the singly charged fragments to consist of protons and deuterons only.

The experimental programme to determine λ_d and f(d, p) would then go as follows. The emulsions can be scanned for nuclear interactions produced by relativistic heavy nuclei. The tracks of singly charged particles emerging from these events can then be scrutinised for further nuclear interactions; amongst these the number of interactions of the (d, p) type can also be estimated. Further the value of λ_d can be evaluated from a knowledge of the interaction mean free path of protons in nuclear emulsions and the proportion of deuterons amongst the singly charged particles. The latter can be taken to be the same as exists amongst singly charged particles emerging from interactions produced by non-relativistic heavy nuclei; this involves the assumption that the processes of fragmentation of heavy nuclei are independent of primary energy for which there is some experimental support. The detailed procedure and results are given in the following section.

[3] G. QUARENI and G. T. ZORN: *Nuovo Cimento*, **1**, 1282 (1955).

3. – Experimental procedure and results.

3·1. *Low energy data.* – From recent investigations on the fragmentation of heavy nuclei ($Z \geqslant 3$) in nuclear emulsions, LOHRMANN and TEUCHER [4] have shown that, for fragment nuclei with $Z \geqslant 2$, there is no detectable dependence of the fragmentation constants on energy from about 1.5 GeV/nucleon up to about 7 GeV/nucleon. We, therefore, assume that the proportion of protons, deuterons and tritons emitted in the fragmentation of heavy nuclei is also independent of energy and proceed to evaluate it from the analysis of singly charged particles arising in the break up of low energy heavy nuclei The interactions were obtained by following the tracks of heavy primary nuclei. in an emulsion stack [5] exposed on a balloon flight from Iowa, U.S.A. ($\lambda = 53^{\circ}$ N). Only those interactions in which *all* tracks had grain densities > 1.4 the values at the plateau of ionization were used. (This was the definition of a low energy interaction.) A total of 27 such interactions were selected for this analysis. From these interactions 36 grey tracks were obtained which were attributed as due to singly charged fragments of the incident heavy nucleus.

The particles producing these grey tracks were identified by the multiple scattering (constant sagitta) *vs.* range method and the grain density *vs.* multiple scattering method; the ratio $p:d:t$ was obtained as 18:15:2. (In addition there is one track which gives a mass value significantly higher than that of a triton and is probably due to an α-particle.) We have used this ratio to determine the interaction mean free path for high energy deuterons.

3·2. *High energy data.* – The results given in this section were obtained from:

a) three stacks exposed near the geomagnetic equator where the vertical cut-off energy is 5 GeV/nucleon;

and *b*) three other stacks exposed over Northern Italy where the vertical cut-off energy is 1.55 GeV/nucleon [6].

[4] E. LOHRMANN and M. W. TEUCHER: *Phys. Rev.*, **115**, 636 (1959).

[5] S. BISWAS, P. J. LAVAKARE, K. A. NEELAKANTAN and P. G. SHUKLA: *Nuovo Cimento* (in course of publication).

[6] The details on the stacks are given by: R. CESTER, A. DEBENEDETTI, C. M. GARELLI, B. QUASSIATI, L. TALLONE and M. VIGONE: *Nuovo Cimento*, **7**, 371 (1958) and by: P. H. FOWLER and C. J. WADDINGTON: *Phil. Mag.*, **1**, 637 (1956). One of us (P. K. A.) wishes to thank Drs. DEBENEDETTI and QUASSIATI of Torino and Drs. PERKINS and WADDINGTON of Bristol for making available the emulsion stacks and the scanning charts.

Singly charged fragments emerging from interactions produced by multiply charged particles were selected from considerations of the angles they make ([1,7]) and were followed from emulsion to emulsion until they produced an interaction or left the stack. (In the same interaction there were usually fragment α-particles or heavier nuclei with respect to which singly charged particles could be followed with ease.)

In a total of 335 cm of track length in stacks of type (*a*), 15 interactions were observed, while 26 interactions were obtained from a following of 566 cm in stacks of type (*b*). The corresponding mean free paths for the fragments were obtained as 22.3 cm in stacks of type (*a*) and 21.7 cm in stack of type (*b*). It is seen that the mean free paths in the two energy regions are almost identical; we have, therefore, combined these results and obtained a value of 21.9 cm for the interaction mean free path for high energy singly charged fragments.

The interaction mean free path of high energy deuterons can now be obtained from the relation:

$$\lambda_d = \frac{F \cdot l \cdot \lambda_p}{n\lambda_p - (1 - F)l},$$

where: λ_p = interaction mean free path of high energy protons in emulsions = 34.7 cm ([1]);

l = total track length in cm followed in the emulsion stack and caused by singly charged high energy fragments;

n = number of secondary interactions;

and $F = \dfrac{d}{p + d} = \dfrac{17}{35}$ (d here incluedes the two tritons).

Using these constants, the value of λ_d in the present experiment is obtained as 15.8 cm. (This may be compared with a value of (14.7 $\pm$ 4) cm obtained by us for deuterons of total energy 200 MeV (*).)

Out of the 41 secondary interactions, 11 were of the (d, p) type. (A cut-off angle of $2 \cdot 10^{-2}$ rad ([1]) was used to separate this type of star from all other interactions in the *a*-type of stacks while an angle of 5° ([7]) was used in the *b*-type of stacks; these (d, p) type interactions have been corrected for the pseudo cases which arise because of secondary shower particles lying within

([7]) C. J. WADDINGTON: *Phil. Mag.*, **2**, 1059 (1957).

(*) We are thankful to Dr. W. H. BARKAS of the University of California, U.S.A. for kindly giving us an emulsion plate exposed to the deuteron beam.

the cut-off angles.) With the above determined values for the interaction mean free path of high energy deuterons and their relative proportion with respect to protons, the number of deuteron induced interactions is estimated to be 27.6. The value of f(d, p) is therefore 0.4.

4. – Limitation of the method.

Appa Rao *et al.* [1] have shown that in interactions produced by α-particles of energy > 5 GeV/nucleon, about 5% of the particles within the cut-off angle ($2 \cdot 10^{-2}$ rad) are « secondary shower particles ». This « contamination » within the cut-off angle sets a limit to the minimum detectable proportion of deuterons which must exist among the singly charged particles of the primary radiation outside the earth's magnetic field, so as to be detectable. This minimum detectable proportion can be estimated as follows:

Let the percentage of deuterons among the singly charged particles *outside the earth's magnetic field* be N. Then, near the geomagnetic equator, where we propose to do the experiment, the difference in the cut-off energies for protons and deuterons, will cause the proportion of deuterons to be about 2.5 times N. If we take the interaction mean free path for deuterons to be about half that for protons, (as obtained in this experiment), then the proportion of deuteron induced stars to proton induced stars in emulsions wil be about $5\,N\%$. If we further use the value of f(d, p) = 0.4 then the identifiable proportion of deuteron stars will be about $2\,N\%$; therefore, to detect deuterons in the cosmic radiation, assuming that we require at least twice as many genuine deuteron induced events as the « contamination » events, we demand that $2\,N \geqslant 10\%$. Thus it seems that the minimum detectable proportion of deuterons among the primary singly charged particles is about 5%.

The expected proportion of deuterons in the cosmic radiation in the vicinity of the earth has been calculated according to two theories for the origin of cosmic radiation.

1) Assuming a chemical composition for the cosmic radiation at the source similar to that attributed to supernovae, Hayakawa *et. al.* [8] have calculated the ratio of deuterons to protons N_d/N_p to be $\leqslant 3 \cdot 10^{-3}\,x$, where x is the amount of matter traversed. If the value of x is taken to be about 1 g/cm^2 (as determined from the ratio N_L/M_M [9]), the proportion of deuterons in the

[8] S. Hayakawa, K. Ito and Y. Terashima: *Suppl. Progr. Theor. Phys.*, **6**, 1 (1958).
[9] M. V. K. Appa Rao, S. Biswas, R. R. Daniel, K. A. Neelakantan and B. Peters: *Phys. Rev.*, **110**, 751 (1958).

primary radiation is three parts in a thousand which is far too small to be detected by this technique. A change of x by more than a factor of five would be needed before primary deuterons can be detected by this method.

2) SINGER [10], on the other hand, has calculated this ratio assuming that *a*), the cosmic radiation originated at the source almost entirely as iron nuclei and *b*), that the composition observed at the top of the atmosphere is due to the breakup of these nuclei in interstellar collisions. He then gets the ratio N_d/N_p as $5.2 \cdot 10^{-2}$ at the top of the atmosphere and therefore about 2% beyond the earth's magnetic field. If this point of view is correct it may just be possible to detect primary deuterons. However, in all these calculations a number of assumptions are used and, therefore, an experimental determination of N_d/N_p is still quite important, if only to set limits on the abundance.

5. – Future experiments.

Once the values of λ_d and $f(p, d)$ are evaluated accurately it will be possible to determine the existence of deuterons in the primary cosmic radiation by flying emulsions under few g/cm² of residual atmosphere at geomagnetic latitudes corresponding to a cut-off energy of about 5 GeV/nucleon. (A very high altitude exposure is necessary to reduce the number of high energy deuterons arising from the breaking up of heavy nuclei.) The emulsions could then be scanned for all stars and the events of the (d, p) type separated out. It will be necessary to remove from this sample those stars which were produced during the ascent of the balloon. For this, a moving plate mechanism can be used in which one emulsion is made to move a small distance relative to the other after the balloon attains the ceiling altitude; only those stars in which tracks are traceable from emulsion to emulsion need be considered. Exposures of this type are being carried out on the current plastic balloon ascents being conducted by the flight group of this laboratory.

* * *

We would like to express our thanks to Professor M. G. K. MENON for reading the manuscript and making useful suggestions. One of us (P. K. ADITYA) would like to take this opportunity to thank Professor NIELS BOHR for the hospitality given to him at the Institute for Theoretical Physics, Copenhagen, on leave from where the mesaurements were made at Turin and Bristol.

(10) S. F. SINGER: *Suppl. Nuovo Cimento*, **8**, 549 (1958).

RIASSUNTO (*)

Si descrive un metodo che può essere usato per rivelare la presenza di deutoni di alta energia nella radiazione cosmica primaria. Il metodo suggerito trae origine dal fatto che quando solo il neutrone del deutone che arriva prende parte ad una collisione e dà origine ad una stella nell'emulsione nucleare, il protone continua in avanti quasi senza deviazione. Queste stelle, chiamate stelle (d, p), possono fornire la prova che la particella che arriva è un deutone. Per usare questo metodo è necessario conoscere: 1) il cammino libero medio di interazione λ_d dei deutoni di alta energia nelle emulsioni nucleari; e 2) la proporzione $f(d, p)$ delle interazioni provocate dai deutoni, a cui prende parte il solo neutrone. Questi valori sono stati determinati in $\lambda_d = 15.8$ cm e $f(d, p) = 0.4$. Il metodo suggerito è applicabile se i deutoni costituiscono $\geqslant 5\%$ delle particelle a carica singola della radiazione primaria.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

Über die Berechnung der Schallgeschwindigkeit in Gasmischungen.

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Zagreb

(ricevuto il 10 Maggio 1960)

Zusammenfassung. — In dieser Mitteilung wird zuerst eine neue Ableitung der Beziehung gezeigt, welche die Schallgeschwindigkeit, das Verhältnis der spezifischen Wärmen und die Dichte der Gasmischung idealer Gase mit enspreschenden Größen einzelner Bestandteile der Mischung durch eine Formel in Zusammenhang bringt. Danach wird das Verhältnis der spezifischen Wärmen einer Gasmischung idealer Gase mit entsprechenden Verhältnissen der spezifischen Wärmen seiner Bestandteile in Beziehung gebracht. Zuletzt wird auf Grund von allen diesen Beziehungen die Formel für die Ausbreitungsgeschwindigkeit des Schalles in Gasmischungen idealer Gase abgeleitet.

In drei Mitteilungen, die anderwärts veröffentlicht wurden (1), wurde gezeigt, daß zwischen den Geschwindigkeiten v_j der Ausbreitung des Schalles in den einzelnen Bestandteilen der Gasmischung und der Schallgeschwindigkeit v in der Mischung selbst eine Beziehung besteht, welche in der Gleichung

$$\frac{\varrho_0 v^2}{k} = \sum_{j=1}^{n} \frac{\varrho_{j,0} v_j^2}{k_j} \tag{1}$$

ausdrückbar ist. Hier bedeuten $\varrho_{j,0}$ die Dichten und k_j die Verhältnisse der spezifischen Wärmen der einzelnen Bestandteile der Mischung, während ϱ_0

(1) *Anzeiger d. Österr. Akad. d. Wiss.*, **12**, 251 (1957); **7**, 121 (1958); **11**, 225 (1959).

und k die Dichte und das Verhältnis der spezifischen Wärmen der Mischung selbst bedeuten.

In dieser Mitteilung wird zuerst eine neue Ableitung der Gleichung (1) ausgeführt und danach das Verhältnis k der spezifischen Wärmen der Mischung auf Grund der Verhältnisse k_j der einzelnen Bestandteile abgeleitet; zuletzt wird gezeigt, wie auf Grund alles dessen die Schallgeschwindigkeit in Gasmischungen bestimmbar ist.

1. – Wenn es sich um eine Mischung idealer Gase handeln würde, welche gegeneinander chemisch indifferent sind, dann bestehen nach L. BOLTZMANN [2] im thermischen Gleichgewicht einzelner Bestandteile in der Gasmischung die Gleichungen (m_j und m_h seien die Molekularmassen einzelner Bestandteile)

$$m_j \overline{c_j^2} = m_h \overline{c_h^2} \qquad (j, h = 1, 2, \ldots, n), \tag{2}$$

wo $\overline{c_j^2}$ und $\overline{c_h^2}$ die Mittelwerte der Geschwindigkeitsquadrate der Molekeln einzelner Bestandteile bedeuten.

Da diese Gleichungen für alle Bestandteile der Gasmischung nach L. BOLTZMANN gültig sein müssen, so soll dies auch für die mittlere Molekularmasse m der Mischung gültig sein, d.h.

$$m \overline{c^2} = m_h \overline{c_h^2} \qquad (h = 1, 2, \ldots, n), \tag{3}$$

wo $\overline{c^2}$ den Mittelwert des Geschwindigkeitsquadrates aller Teilchen der Mischung bezeichnet.

Infolge der bekannten Formel [3] für die Berechnung der mittleren Molekularmasse der Gasmischung

$$m = \frac{\varrho_0}{\sum_{j=1}^{n} \frac{\varrho_0}{m_j}} \tag{4}$$

geht die Boltzmannsche Beziehung (3) in die Gleichung über

$$\varrho_0 \overline{c^2} = m_h \overline{c_h^2} \sum_{j=1}^{n} \frac{\varrho_{j,0}}{m_j} \tag{5}$$

[2] Vgl. z.B.: C. SCHAEFER: *Einf. Theor. Phys.*, **2**, 351 (1921).
[3] Vgl. z.B.: R. SEELIGER: *Auf. Theor. Phys.*, **51-52**, 128 (1921).

und diese wieder mit Rücksicht auf (2) in

$$\varrho_0 \overline{c^2} = \sum_{j=1}^{n} \varrho_{j,0} \overline{c_j^2} \,. \tag{6}$$

Diese Gleichung kann als eine Art Definition des mittleren Geschwindigkeitsquadrates der Gasmischung idealer Gase dienen.

Mittels der Anwendung der bekannten Relation (1), die die Beziehung zwischen dem mittleren Geschwindigkeitsquadrat der Teilchen und der Ausbreitungsgeschwindigkeit c des Schalles ergibt, erhält man zuletzt aus (6) die Gleichung (1).

Wie aus dieser Ableitung der Gleichung (1) ersichtlich, wurde sie unter dem BOLTZMANN'schen Postulat für das thermische Gleichgewicht verschiedener Gase in der Gasmischung durchgeführt.

2. – Um die Anwendung der Gleichung (1) zur Berechnung der Ausbreitungsgeschwindigkeit des Schalles in den Gasmischungen zu ermöglichen, ist es notwendig, das Verhältnis k der spezifischen Wärmen mittels der Verhältnisse der spezifischen Wärmen einzelner Bestandteile der Mischung zu bestimmen.

Zu diesem Zweck soll die bekannte Formel (5) $k = 1 + (2/f)$ bzw.

$$f = \frac{2}{k-1} \tag{8}$$

angewendet werden. Im Einklang mit dem Postulat (6) von L. BOLTZMANN, nach welchem sich die mittleren kinetischen Energien zweier Gase (die sich im thermischen Gleichgewicht befinden) wie die Zahl der Freiheitsgrade derselben gegeneinander beziehen, lassen wir auch hier die Zahl der Freiheitsgrade der Mischung mit dem Verhältnis der mittleren kinetischen Energien der betreffenden Gase beurteilen. Wir definieren also

$$f = \frac{\sum_{j=1}^{n} f_j \varrho_{j,0} \overline{c_j^2}}{\sum_{j=1}^{n} \varrho_{j,0} \overline{c_j^2}} \tag{9}$$

(1) J. M. JEANS und R. FÜRTH: *Dynamische Theorie der Gase* (1926), p. 156.
(5) Vgl. z.B.: C. SCHAEFER: *Einf. Theor. Phys.*, **2**, 353 (1921).
(6) Vgl. z.B.: C. SCHAEFER: *Einf. Theor. Phys.*, **2**, 349 (1921).

und können gleich daraus schließen, daß f im allgemeinen keine ganze Zahl sein wird (obzwar die Freiheitsgrade f_j der Teilchen einzelner Bestandteile ganze Zahlen sind). Dies geht klar hervor daraus, weil wir, die Mischung mittels beliebiger Wahl der Mengen einzelner Bestandteile beliebig ändern können; falls ausnahmsweise alle Freiheitsgrade f_j der Bestandteile der Mischung untereinander gleich würden, so würde auch $f = f_j$ bestimmt eine ganze Zahl sein. Man soll also die Zahl f der Freiheitsgrade der Mischung mehr als eine mathematische Hilfsgröße betrachten, welche uns nur zur Ausgleichung der beiden Seiten der Gleichung (9) dient.

Durch Anwendung des Ausdrucks für f aus (8) auf beide Seiten der Gleichung (9) erhält man

$$k - 1 = \frac{\sum_{j=1}^{n} \varrho_{j,0} \overline{c_j^2}}{\sum_{j=1}^{n} \varrho_{j,0} \overline{c_j^2} k_j - 1} . \tag{10}$$

Da aber für ideale Gase bekannterweise die Relation (7)

$$\overline{c_j^2} = 3 \frac{R}{m_j} T_j \qquad (j = 1, 2, \ldots, n) \tag{11}$$

gültig ist, so ergibt sich aus (10), unter der Beachtung, daß in der Mischung $T_j = T$ wird, zuletzt

$$k - 1 = \frac{\sum_{j=1}^{n} \frac{\varrho_{j,0}}{m_j}}{\sum_{j=1}^{n} \frac{\varrho_{j,0}}{m_j} \cdot \frac{1}{k_j - 1}} , \tag{12}$$

eine Formel, welche auch von B. Galeb (8) auf andere Weise abgeleitet wurde.

Durch Einsetzen dieses Ausdrucks für k in (1) kommt man zuletzt für die Ausbreitungsgeschwindigkeit v des Schalles in Gasmischungen idealer Gase zu der Gleichung

$$v^2 = \left(1 + \frac{\sum_{j=1}^{n} \frac{\varrho_{j,0}}{m_j}}{\sum_{j=1}^{n} \frac{\varrho_{j,0}}{m_j} \cdot \frac{1}{k_j - 1}} \right) \sum_{j=1}^{n} \frac{\varrho_{j,0} v_j^2}{\varrho_0 k_j} . \tag{13}$$

(7) G. Jäger: *Die Fortschritte der kinetischen Gastheorie* (1919), p. 65.
(8) Nach mündlicher Mitteilung.

Die Formel (13) scheint es zu ermöglichen, wenigstens die angenäherte Schallgeschwindigkeit in denjenigen Planetatmosphären zu berechnen, welche aus Gasmischungen bestehen, falls uns die chemischen und physikalischen Eigenschaften derselben bekannt sind und die Anwendung der Gleichung (1) gestatten.

RIASSUNTO (*)

Nella presente comunicazione si mostra prima una nuova derivazione della formula che collega la velocità del suono, il rapporto dei calori specifici e la densità del miscuglio. Si mettono poi in relazione fra di loro il rapporto dei calori specifici di un miscuglio gassoso di gas ideali con i corrispondenti rapporti dei calori specifici dei suoi componenti. Sulla base di tutte queste relazioni si ricava infine la formula per la velocità di propagazione del suono nei miscugli gassosi di gas ideali.

(*) *Traduzione a cura della Redazione.*

Theory of Fusion Reactions in an Unconfined Plasma.

J. G. Linhart

EURATOM - Bruxelles

(ricevuto l'11 Maggio 1960)

Summary. — A criterion for a zero energy gain fusion reactor is obtained, the reactor resembling a cylindrical H-bomb in which a plasma is either not confined or only marginally confined. A magnetically driven cylindrical plasma shell is shown to generate a plasma column for which the above mentioned criterion could be satisfied.

1. – Introduction.

It has been emphasized that in order to derive a power gain from thermonuclear reaction in plasma a long-term confinement of a volume of plasma is required [1]. A number of proposals, aiming at the achievement of such confinement, has been considered during the past ten years, all of which appear to be inadequate owing to the instability of confinement [2-4] and owing to large radiation losses [5-7]. These drawbacks may be overcome in the future and several lines of approach have been suggested [8-10].

[1] J. D. Lawson: *Proc. Phys. Soc.*, **70***b*, 5-10 (1957).

[2] T. Coor, S. P. Cunningham, R. A. Ellis, M. A. Heald and A. Z. Kranz: *Proc. 2nd Int. Conf. Peaceful Uses of At. En.*, 32, 201.

[3] W. B. Thomson *et al.*: *Proc. 2nd Int. Conf. Peaceful Uses of At. En.*, 32, 65.

[4] L. C. Burkhardt *et al.*: *Journ. Appl. Ph.*, **29**, 64-67 (1958).

[5] J. Tuck: *Proc. of 4th Int. Conf. on El. Disch. in Gases*, Uppsala (1959).

[6] J. Drummond: *Proc. Inst. for Plasma Phys.*, Univ. of Washington (1959).

[7] J. Drummond and M. Rosenbluth: *Proc. Inst. for Plasma Phys.*, Univ. of Washington (1959).

[8] E. Persico and J. G. Linhart: *Nuovo Cimento*, **8**, 740 (1958).

[9] E. S. Weibel *et al.*: *Proc. 2nd Int. Conf. on Peaceful Uses of Atomic Energy*, 32, 161 (1958).

[10] N, Christofilos: *Proc. 2nd Int. Conf. on Peaceful Uses of Atomic Energy*, 32, 279 (1958); G. J. Budker: *CERN Symposium Geneva* (1956); J. G. Linhart and A. Schoch: *Nucl. Instr. and Methods*, **4**, 332 (1959).

In the case of a long-term confinement not being discovered or such confinement leading to a technically useless apparatus, a solution to the liberation of fusion energy can be sought in systems exhibiting only a marginal confinement or even no confinement. Such system are akin to H-bombs, the use of which for power production has been suggested elsewhere [11]. Here we shall analyse a small version of an unconfined plasma reactor whose power output can be absorbed by laboratory apparatus. Such a line of approach to the problem of controled fusion reactor has been mentioned independently by ARTSIMOVICH and TUCK during the 1958 Conference on Peaceful Uses of Atomic Energy.

2. – The principle of an unconfined fusion reactor.

Let us consider an infinitely long plasma column of radius r_0 in which the plasma density and temperature are initially uniform (Fig. 1). The plasma is composed of equal parts of deuterons and tritons and a corresponding number of neutralising electrons. In absence of any constraining force the column will expand radially. However, let us remember that the electron gas is spatially coupled to the positive ion gas by electric fields and is, therefore, constrained in its expansion by the inertia of the positive ion gas. At the same time the collisions between electrons and positive ions cause the difference in temperature between the two gases to be small as compared with the average plasma temperature. It thus follows that the random velocities of the particles in the plasma are steadily converted in the energy of radial plasma flow away from the axis. This behaviour can be expressed approximately by an equation of state

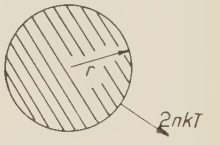

Fig. 1. – A simple model for an expanding cylindrical column of plasma.

$$\frac{T}{T_0} = \left(\frac{r_0}{r}\right)^{\frac{4}{3}} \qquad \text{where} \qquad T_e = T_p = T \tag{1}$$

and an equation of motion

$$NM\ddot{r} = 2\pi r kT n\,, \tag{2}$$

[11] E. TELLER: *Proc. of 2nd Int. Conf. on Peaceful Uses of Atomic Energy*, 31, 32.

where n is the density and N is the linear density of positive ions, M is the mean ion mass. Substituting from eq. (1) into eq. (2) it follows

$$\ddot{r} = \frac{2kT_0}{M} \frac{r_0^{\frac{1}{3}}}{r^{\frac{1}{3}}} \tag{2a}$$

or

$$\dot{r} = \left\{ \frac{3kT_0}{M} \left[1 - \left(\frac{r_0}{r} \right)^{\frac{4}{3}} \right] \right\}^{\frac{1}{2}} . \tag{2b}$$

From this it follows that after an expansion of the plasma column to $r = 4r_0$ the velocity of the radial flow reaches about 90% of the maximum attainable value. The original step distribution of plasma density will diffuse, during the expansion, into a more gradual one (Fig. 2). We have neglected this feature in our approximate model; however, this does not alter our main conclusions about the expansion process.

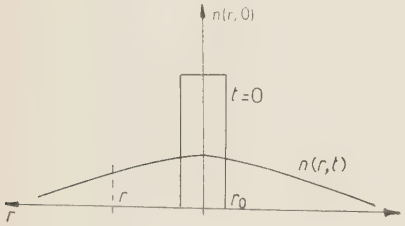

Fig. 2. – The initial density distribution $n(r, 0)$ and its development after a time t.

Let us now calculate the energy W_F generated by DT reactions during the expansion of the plasma. This can be written formally as

$$W_F = \tfrac{1}{2} N \cdot P \cdot W_0 , \tag{3}$$

where P is the probability that a deuteron will be involved in a DT reaction during the expansion. W_0 is the energy released in a single reaction and is $2.8 \cdot 10^{-5}$ erg.

P can be expressed using the expression $\langle w\sigma_n \rangle$ for the mean value of the rate of the DT reaction. Thus

$$P = \int_{r_0}^{\infty} \frac{n}{2} \langle w\sigma_n \rangle \frac{\mathrm{d}r}{\dot{r}} . \tag{4}$$

If the mean random speed of the D and T nuclei correspond to an energy of about 30 KeV or more the averaging process required for the formation of $w\sigma_n$ can be simplified because in this region σ_n is not a very steep function of w. Thus we may write

$$\langle w\sigma_n \rangle \sim \sqrt{\frac{2kT}{M}} \, \sigma_n(T) .$$

In this case equation (4) can be written as

$$P = \frac{1}{2}\int_{r_0}^{\infty} \sqrt{\frac{2kT(r)}{M}}\,\frac{N}{\pi r^2}\,\sigma_n[T(r)]\,\frac{\mathrm{d}r}{\dot{r}}\,. \tag{4a}$$

or using equation (2*b*) and putting $X = r/r_0$

$$P = \frac{1}{2}\sqrt{\frac{2}{3}}\,\frac{N}{\pi r_0}\int_1^{\infty} X^{-\frac{8}{3}}(1 - X^{-\frac{4}{3}})^{-\frac{1}{2}}\sigma_n\,\mathrm{d}X\,. \tag{4b}$$

The function $\sigma_n(T)$ has the form shown in Fig. 3. For the purpose of evaluation of the integral we shall use a formula which represents approximately the curve of Fig. 3. Thus

$$\sigma_n \simeq \sigma_m \exp\left[\left(-\frac{T_m - T}{0.43\,T_m}\right)^2\right]\ (\mathrm{cm}^2),$$

where $T_m = 10^9$ (°K) and $\sigma_m = 5\cdot 10^{-24}$ (cm²).

Let us now assume that T_m is the same as T_0 (*). Then equation (4*b*) becomes

$$P = \frac{1}{2}\sqrt{\frac{2}{3}}\,\frac{N\sigma_m}{\pi r_0}\int_1^{\infty} X^{-\frac{8}{3}}[1 - X^{-\frac{4}{3}}]^{-\frac{1}{2}}\cdot$$

$$\cdot \exp[-5.5(X^{\frac{4}{3}} - 1)^2]\,\mathrm{d}x\,.$$

Putting $-X^{-\frac{3}{4}} + 1 = y$, $\mathrm{d}x = \frac{3}{4}\cdot$ $\cdot \mathrm{d}y(1+y)^{-\frac{7}{4}}$ one has

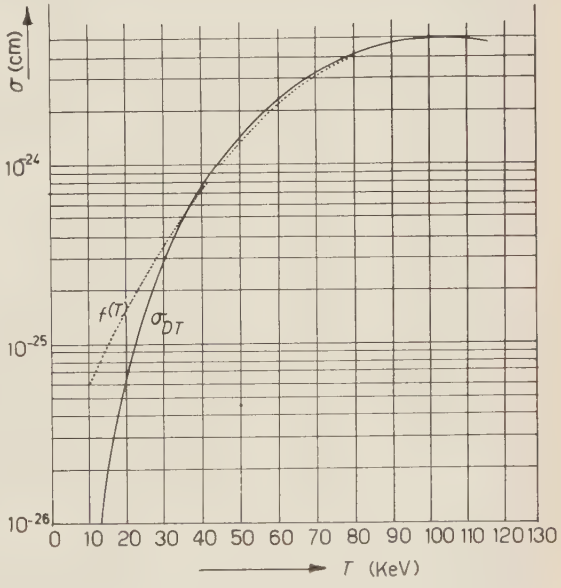

Fig. 3. – The function $f(T)$ approximating $\sigma_{DT}(T)$.

$$P = \frac{1}{2}\sqrt{\frac{3}{8}}\,\frac{N\sigma_m}{\pi r_0}\int_0^1 y^{-\frac{1}{2}}(1 - y)^{\frac{1}{2}}\exp[-5.5y^2]\,\mathrm{d}y\,. \tag{4c}$$

It is evident that the biggest contribution to the integral will lie within

$$0 < y < 0.43\,.$$

(*) This represents an approximate condition for an optimum operation of the unconfined reactor.

The value of the integral is, therefore, approximately

$$\int_0^{0.43} \frac{0.7}{\sqrt{y}}\,\mathrm{d}y\ .$$

Thus

$$P \simeq 0.1\,\frac{N\sigma_m}{r_0} \tag{4d}$$

and the expression for the fusion output becomes

$$W_F = 0.7\cdot 10^{-29}\,\frac{N^2}{r_0}\ \text{(erg)}\ (^*)\ . \tag{5}$$

The loss of energy per unit length (W_L) is readily computed if one assumes that the expansion results in a complete loss of all the kinetic and thermal energy of electrons and positive ions. Thus

$$W_L = 3N\,kT_0 = 4.5\cdot 10^{-7}\cdot N\ . \qquad \text{(erg)} \tag{6}$$

The condition for a net energy gain is

$$W_F > W_L \tag{7}$$

and therefore, using equations (5) and (6) one obtains

$$\frac{N}{r_0} > 0.6\cdot 10^{23}\ . \tag{8}$$

Thus, if the linear density is large enough to satisfy this inequality, an energy gain can be expected from a completely unconfined plasma configuration.

The requirement of a large N is embarrassing technically owing to the enormous stored energy per unit length. This energy is the same as W_L calculated in eq. (6). Let us consider a zero-energy-gain reactor. The inequality (8) then becomes an equation, calculating N from this equation and substituting it in eq. (6) we get for the stored energy of a zero-gain reactor

$$W_S = W_L = \tfrac{1}{4}\cdot 10^{10} r_0\ \text{(Joules)}\ . \tag{9}$$

(*) A somewhat higher value would have been obtained were it not assumed that $\langle \sigma_n w\rangle = \sqrt{2kT/M}\,\sigma_n$.

Comparing this with the energy liberated in an explosion of one ton of TNT, which is about 10^{10} Joules, it is evident that his type of reactor is somewhere between a large rocket engine and a small atom bomb.

There are at least three ways in which to minimize the requirement for such high stored energies. One follows directly from eq. (9), *i.e.*, an attempt should be made to decrease r_0 as much as possible. The second is to try to decrease the expansion speed below that of free expansion (eq. (2*b*)). This implies the use of some confinement mechanism (*). The third, and most important one, is connected with the second. One may try to recuperate the energy of the radial plasma flow during the expansion of the plasma column.

3. – An imploding hollow shell of plasma.

All the three above-mentioned objectives may be accomplished by using a magnetic field to drive a hollow plasma shell.

Let us now describe a simple model of such a configuration. A cylindrical shell of plasma of effective thickness d is driven towards the axis of symmetry with an acceleration

$$\ddot{r} = \frac{B_\varphi^2}{8\pi n M d} ; \tag{10}$$

where nMd is the plasma mass per cm² of the surface of the shell (Fig. 4).

Obviously

$$N = 2\pi r n d \ \text{(el/cm)}$$

and

$$B_\varphi = \frac{2I}{r} ,$$

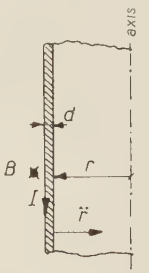

Fig. 4. – A model of a collapsing cylindrical plasma shell.

where I is the total current in plasma. Eq. (10) can, therefore, be rewritten as

$$\ddot{r} = \frac{I^2}{rMN} . \tag{10a}$$

This shows that in a current-fed device the acceleration will increase as r decreases. However, in a discharge connected to a capacitor bank, I will decrease when r decreases below a certain value. Let us, therefore, assume a nearly constant acceleration. In such a uniformly accelerated system the

(*) A paper analising this possibility is in press.

plasma will approach the Halley density distribution whose effective thickness is

$$d = \frac{kT}{M\ddot{r}} = \frac{4kTN}{rB_q^2} \tag{11}$$

provided $d \ll r$ (*).

The plasma temperature T will be dependent on the initial temperature T_1, on the ohmic loss due to the flow of current I and on the change in the thickness d and in the radius r.

If one can neglect the mixing of the magnetic field and plasma (**), one may consider an increase in T to be the result of the compression of the shell rather than being due to the ohmic loss. In that case the temperature T and the thickness d follows from eq. (11) and an equation of state

$$\frac{T}{T_1} = \left(\frac{r_1 d_1}{rd}\right)^{\frac{8}{3}} . \tag{12}$$

Thus

$$\left(\frac{T}{T_1}\right)^{\frac{5}{4}} = \frac{B\varphi}{B\varphi_1} , \tag{13a}$$

where T_1 and $B\varphi_1$ are evaluated after the formation of the plasma shell at the radius r_1. Similarly

$$\frac{d_1}{d} = \frac{r}{r_1}\left(\frac{B\varphi}{B\varphi_1}\right)^{\frac{6}{5}} . \tag{13b}$$

The ratio $B\varphi/B\varphi_1$ can be of the order of 100, whereas one hopes that r_1/r will be of the order of 1000. It thus follows that d_1/d will be about $\frac{1}{4}$.

It will be appreciated that the thickness d can be very small; if, *e.g.* $\dot{r}$ approaches $2 \cdot 10^8$ cm/s near the axis and if the radial distance over which the collapse occurs is 50 cm, then

$$\langle \ddot{r} \rangle = 4 \cdot 10^{14} \text{ cm/s}^2 .$$

If, in the same time, T_1 is about 10^5 (°K), the thickness d of a hydrogen plasma near the axis becomes $d \sim 0.04$ cm.

(*) If the acceleration increases with decreasing r, as suggested by eq. (10*a*), we would expect a lower value for d than that given by eq. (11). If the model of a Halley distribution is to apply, the acceleration must not change too rapidly (Appendix I).

(**) This effect boarders on the problem of dynamic stability of the accelerated plasma shell. This will be treated in a separate publication and shown not to be of great importance in our case.

Let us now describe the formation of the plasma on the axis of the system. This occurs as a result of collisions within the hollow shell which for $r < d$ is collapsing on itself. Let us restrict ourselves to the case of such large plasma densities that the nuclei in the intershocking plasma become nearly thermalized. The criterion for this case is (*)

$$\frac{N}{d} > 0.5 \cdot 10^{22} . \tag{14}$$

a plasma column formed in this way corresponds to the model used in deriving the reactor equation (8).

The minimum diameter of the plasma column can be deduced as follows (**). After thermalization only $\frac{1}{6}$ of the original energy W_k of an average ion will be associated with its radial speed, $\frac{1}{2}W_k$ having been transfered to the electron gas and $\frac{2}{6}$ being associated with its other two degrees of freedom. Thus its radial speed will be only $\sqrt{\frac{1}{6}}$ of the maximum speed the ion gained before the shock took place. The minimum diameter of the plasma column will be, therefore, approximately $d/\sqrt{6}$.

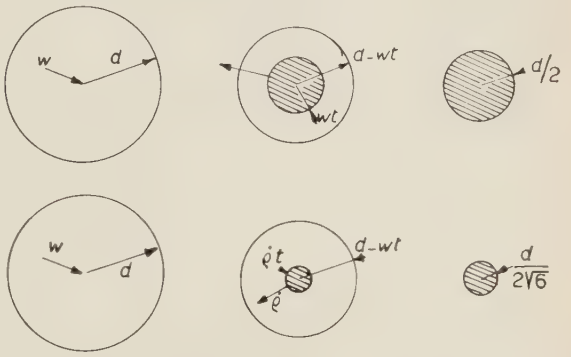

Fig. 5. – Development of the minimum radius column without collisions and with full thermalization.

The two extreme cases of the development of the thinnest plasma column are shown in Fig. 5. In the case of no collisions the minimum radius r_0 is $d/2$ whereas in the shock generated by collisions and satisfying the inequality (13) the minimum radius r_0 is $d/(2\sqrt{6})$. This value can be substituted into inequality (8) to give a criterion for the zero-energy gain reactor

$$\frac{N}{d} > 0.12 \cdot 10^{23} . \tag{15}$$

After the collapse to a minimum radius, the plasma will expand and encounter now a decelerating force of the pinching magnetic field $B\varphi$. Such a

(*) See Appendix II. A less stringent criterion would be obtained if one took into account the electron-nuclei collisions. However, these depend on the temperature of the electron gas and will not be considered.

(**) For a rigorous derivation see Appendix III.

deceleration cannot, however, alter substantially the results reached in eq. (2*b*) and therefore, if a useful confinement of the plasma were required a different mechanism would have to be used. On the other hand, it is obvious that the energy of the expanding shell will be transformed back into the magnetic field $B\varphi$ at the rate of $IB\varphi\dot{r}$ and, through this, into the stored electric energy of the source of the plasma current I.

The efficiency of such a recuperation mechanism depends on many factors and can be estimated probably only with the help of an experiment.

Neglecting radiation losses and loss in solid conductors the efficiency can be made as close to 100% as desired, provided the expansion ratio is made correspondingly large ([12]). The radiation losses are probably small owing to the rapidly decreasing electron temperature.

Let us assume that the total energy loss per one half cycle, *i.e.*, during a single collapse and expansion is π/Q of the total stored energy. Using eq. (14) the criterion (7) can be written as

$$\frac{N}{d} > 0.38\,\frac{10^{23}}{Q}\,. \tag{15a}$$

As an example let us put $d = 0.1$ cm, $Q = 10$. Then

$$N > 3.8\cdot 10^{20}\,.$$

The stored energy in the column is then

$$W_s = 3kTN = 1.5\cdot 10^{7}\ \text{(J/cm)}\,.$$

The energy dissipated during the half-cycle will be

$$W_L = 4.7\,M\ \text{J/cm}\,.$$

The absorption of such an amount of energy by a reactor chamber whose radius is of the order of 100 cm seems to be within the scope of most materials likely to be used in the construction of such apparatus.

4. – Conclusion.

The mechanism described in Section 3 may not be the only one, or even the one, which would make the realization of miniature power-producing H-bombs possible. In the first place it may be necessary to use a toroidal

([12]) J. G. LINHART: *Plasma physics* (Amsterdam, 1960), pp. 223-224.

system, rather than a cylindrical one. Secondly the requirement of very large stored energy may lead one to the use of hydromagnetic capacitors ([13]) rather than conventional ones and thirdly some partial confinement near r_0 may be necessary. However, even without such modifying features, it appears that these miniature bombs are not out of bounds of our technology.

Appendix I

The force $i_z B$ is a force acting on the outside layer of the accelerated shell only. An increase of this force by $\mathrm{d}F$ in a time $\mathrm{d}t$ will be transmitted to a layer whose thickness is

$$\mathrm{d}\varrho = \sqrt{\frac{2kT_s}{M}} \cdot \mathrm{d}t \,.$$

Our assumption of a definite Halley's distance

$$d = \frac{k T_s}{\ddot{R} M}$$

is applicable only if the force F does not change so rapidly that $\ddot{R}$ at the boundary is appreciably different from $\ddot{R}$ felt by particles at a distance d away from it. Thus one requires that

$$\frac{\Delta \ddot{R}}{\ddot{R}} \ll 1 \,,$$

where $\Delta \ddot{R}$ is the change in $\ddot{R}$ during a time

$$\Delta t = \frac{d}{\sqrt{2kT_s/M}} = \frac{\sqrt{kT_s/2M}}{\ddot{R}}$$

Thus the criterion for the formation of a quasi-stationary Halley distribution is

$$\frac{\ddot{R}\sqrt{kT_s/2M}}{\ddot{R}^2} \ll 1 \,.$$

([13]) O. Anderson, W. R. Baker, A. Bratenhal, H. P. Furth and W. B. Kunkel: *Journ. Appl. Ph.*, **30**, 188 (1959).

Appendix II

In order that a cylindrical shock of radial thickness d described in Section 3 should be thermalized near the axis, one requires

$$\frac{\langle(\Delta w)^2\rangle(d/2w)}{(2w)^2} > 1 ,$$

where $\langle(\Delta w_\perp)^2\rangle$ is the mean square of the rate of change of velocity of a nucleus at right angles to w, this change being caused by collisions between this nucleus and other nuclei ([14]).

This can be shown to reduce to

$$\frac{\pi e^4 d}{M^2 w^4} n \ln \Lambda > 1 \qquad \text{where } \ln \Lambda \sim 20.$$

For $r = d/2$ it follows that

$$\frac{N}{d} > \frac{1}{4}\frac{M^2 w^4}{e^4 \ln \Lambda} ,$$

which can be also written as

$$\frac{N}{d} > \frac{1}{80}\left(\frac{w}{c}\right)^4 r_D^{-2}$$

where r_D is the classical radius of deuteron, *i.e.*, $r_D = (e^2/Mc^2) = 0.7 \cdot 10^{-16}$ (cm) and thus

$$\frac{N}{d} > 2.6 \cdot 10^{30}\left(\frac{w}{c^4}\right) .$$

As w would correspond to an energy of approximately 100 keV, *e.g.*, for a deuteron $w \sim 2 \cdot 10^8$ cm/s, one obtains finally

$$\frac{N}{d} > 0.5 \cdot 10^{22}$$

Comparing this relation with relation (8) it follows that a system whose N/d is an order of magnitude lower than that required for a zero energy-gain reactor will already produce a nearly thermalized nuclear gas column in the axial region. This could have been anticipated owing to the DT reaction cross-section being two orders of magnitude smaller than the Coulomb cross-section for large angle scattering at 100 keV.

([14]) L. Spitzer: *Physics of fully ionized gases* (New York, 1955), p. 72.

Appendix III

The development of the cylindrical shock.

A rigorous analysis of the development of central plasma from a hollow plasma shell collapsing on itself results in the formulation of a set of two Boltzmann equations together with the third Maxwell equation. This set of non-linear partial differential equations is insoluble by analytical methods. Instead of solving such a set by numerical calculation it appears of more value to choose a simpler physical model for the process than that on which the Boltzmann's description is based. Such a model is readily found if one assumes that the criterion (14) is satisfied. One may then assume that the nuclei, which have collided with the part of the plasma shell moving in the opposite direction, form instantly a core of thermalized plasma, whose density is n', temperature T and radius ϱ. This core is being bombarded by the nuclei of the collapsing plasma shell. These bombarding nuclei are, of course, thermalized on impact and become new members of the plasma in the core (Fig. 6). The equation of motion for the core-plasma becomes:

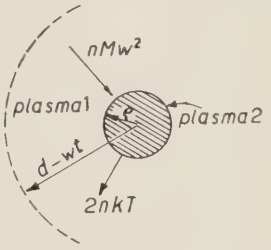

Fig. 6. – A fully thermalised plasma core bombarded by the nuclei of the collapsing plasma shell.

$$\text{(A-3.1)} \qquad \frac{\pi}{2}\,\varrho^2 n' M\ddot{\varrho} = 2\pi\varrho\cdot 2n' kT - 2\pi\varrho n M w^2 \,.$$

Let us take the inflow

$$A = 2\pi r n(r) w$$

to be a constant in a time interval $0 < t < (d-\varrho)/w$. Then $\pi\varrho^2 n' = At$ and eq. (A-3.1) can be written as:

$$\text{(A-3.2)} \qquad \ddot{\varrho} = \frac{8kT}{M}\,\varrho^{-1} - 2\,\frac{w}{t} \,.$$

Neglecting the radiation losses, the temperature T of the plasma in the core is given by

$$\text{(A-3.3)} \qquad \frac{\mathrm{d}W_T}{\mathrm{d}t} = A\cdot\frac{1}{2}\,Mw^2 - 2\pi\varrho n' kT\,\dot{\varrho} \,,$$

where W_T is the heat energy content of the core and is

$$\text{(A-3.4)} \qquad W_T = 3\pi\varrho^2 n' kT = 3A\,kTt \,.$$

With this, eq. (A-3.3) becomes

$$\frac{\mathrm{d}}{\mathrm{d}t}(tT) = \frac{1}{6}\frac{M}{k}w^2 - \frac{2\dot{\varrho}}{3\varrho}(tT)\,. \tag{A-3.5}$$

Equations (A-3.2) and (A-3.5) can be written as

$$\ddot{\varrho}\varrho t = 4\alpha S - 2w\varrho\,, \tag{A-3.6}$$

$$\dot{S} = \frac{w^2}{3\alpha} - \frac{2}{3}\frac{\dot{\varrho}}{\varrho}S\,, \tag{A-3.7}$$

where $2k/M = \alpha$ and $S = Tt$.

From eq. (A-3.6) one has for S

$$S = (4\alpha)^{-1}\varrho(\ddot{\varrho}t + 2w) \tag{A-3.8}$$

and substituting this into eq. (A-3.7) one obtains

$$\tfrac{5}{3}\dot{\varrho}(\ddot{\varrho}t + 2w) + \varrho(\dddot{\varrho} + \ddot{\ddot{\varrho}}t) = \tfrac{4}{3}w^2\,. \tag{A-3.9}$$

It thus follows that the development of the core depends only on w, the speed of the shock. When $t = 0$ one has $\varrho = 0$. Substituting these into eq. (A-3.2) and requiring that $\ddot{\varrho}$ should be finite one gets

$$\dot{\varrho}(t = 0) = \frac{4kT}{Mw}\,. \tag{A-3.10}$$

Substituting this result into eq. (A-3.5) one obtains for the initial temperature

$$T_{(t=0)} = \frac{M}{10k}w^2\,. \tag{A-3.11}$$

Combining equations (A-3.10) and (A-3.11) one has for $(t = 0)$ in terms of the shock speed w

$$\dot{\varrho}(t = 0) = 0.4w\,. \tag{A-3.12}$$

A solution of eq. (A-3.9) is, therefore,

$$\varrho = 0.4wt\,. \tag{A-3.13}$$

According to this result the central core is expanding radially at a constant rate given by eq. (A-3.12). The duration of this expansion is limited by the duration τ of the bombardment of the central core by the collapsing plasma shell. This is

$$\tau = \frac{d - \varrho}{w} \tag{A-3.14}$$

from which

$$\tau = \frac{d}{w - \dot{\varrho}} .$$

During this time the plasma core will have expanded to a radius:

$$\varrho \equiv r_0 = \dot{\varrho}\tau .$$

Thus

(A-3.15) $$r_0 = 0.286\, d$$

which is substantially the same as the r_0 obtained in the main text, where it was found that

$$r_0 \simeq \frac{d}{2\sqrt{6}} = 0.204\, d .$$

RIASSUNTO (*)

Si ottiene un criterio per un reattore di fusione a guadagno di energia zero, reattore che somiglia ad una bomba H cilindrica in cui il plasma o non è costretto o è solo marginalmente costretto. Si mostra che uno strato cilindrico di plasma provocato magneticamente genera una colonna di plasma per cui il criterio suddetto può essere soddisfatto.

(*) *Traduzione a cura della Redazione.*

Lee Model with Single Boson Oscillator.

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(ricevuto il 23 Maggio 1960)

Summary. — We consider a Lee model with only one boson field oscillator. The Heisenberg equations turn out to be linear, and the problem is soluble in all sectors. The spectrum is entirely and essentially discrete, and the physically interesting questions relate not to scattering but to straightforward energy measurements. In particular all the consequences of introducing an indefinite metric can be traced out explicitly; it appears that no consistent physical interpretation can be given.

1. – Introduction.

The Lee model ([1-3]) is too well known to require a detailed introduction. The equations describing the model are explicitly soluble in the lowest non-trivial sector, and exhibit an upper limit on the renormalized coupling constant, or in other words on the result, say, of a measurement of the zero energy N-θ scattering cross-section. If one tries nevertheless to adapt the theory to an experimental situation in which this cross-section exceeds its critical value, then one finds that the Hamiltonian has become non-Hermitian in the process. Attempts to save the situation by introducing an indefinite metric with respect to which the Hamiltonian is at least self-adjoint have not so far been successful.

One of the tantalizing features of the problem is that the basic difficulties of physical interpretation arise only in the higher sectors in which no explicit solution is available. It is proposed therefore to look at a simplified version

(*) On leave of absence from Christ Church, Oxford, England.

([1]) T. D. Lee: *Phys. Rev.*, **95**, 1329 (1954).

([2]) G. Källén and W. Pauli: *Mat.-Fys. Medd.*, **30**, No. 7 (1955).

([3]) W. Heisenberg: *Nuclear Physics*, **4**, 532 (1957).

of the model with only a single boson field oscillator. This is, of course, just a special case of the full Lee model, obtained by making the cut-off a δ-function

The proposed model has two advantages. First, there are just two energy eigenvectors in each sector, while for any model with more than one oscillator the number of states increases with the sector number. It is this that makes the equations soluble in every sector. Second, we have an entirely discrete spectrum, so that the physical quantities of interest are just energy levels and expectation values, and the only measurements performed are energy measurements. Thus the whole discussion takes place at a more elementary level than for the full Lee model in which the S matrix plays the leading role. In view of the near-unanimous conclusion that the metric cannot consistently be made indefinite, it is of some interest to exhibit this inconsistency in as elementary a manner as possible.

2. – Solution of the equations.

2·1. *Schrödinger picture.* – The Hamiltonian is

$$H = (m_V + \delta m)\bar{V}V + m_N\bar{N}N + \mu\bar{\theta}\theta + g(\bar{V}N\theta + \bar{N}V\bar{\theta})$$

with the commutation rules

$$(2.1)\quad [\bar{V}, V]_+ = 1 = [\bar{N}, N]_+ ;\ \ [V, V]_+ = 0 = [N, N]_+ ;\ \ [V, N]_+ = 0 ;\ \ [\bar{\theta}, \theta] = 1 .$$

The operators $(\bar{V}V + \bar{N}N)$ and $(\bar{N}N - \bar{\theta}\theta)$ commute with H. We shall be concerned exclusively with the case where only one fermion is present: $(\bar{V}V + \bar{N}N) = 1$. Denoting the state with n_1 bare V-quanta, n_2 bare N-quanta and n_3 bare θ-quanta by $|n_1, n_2, n_3\rangle$, we see that these bare states interact only in pairs: $|1, 0, n\rangle$ and $|0, 1, n+1\rangle$ form the n-th sector. The states $|0, 0, 0\rangle$, $|0, 1, 0|$ and $|0, 0, 1\rangle$ are simultaneous eigenstates of the uncoupled and of the full Hamiltonians and will not concern us further except insofar as they furnish experimental values for the quantities μ and m_N. (Note also that all no-fermion and two-fermion states are effectively uncoupled.)

In the present model it is more convenient in the long run not to introduce coupling constant- and wave-renormalization. (We shall comment briefly on the renormalized form of the equations in the Appendix.) To simplify the algebra we shall assume $m_N = 0 = m_V$, and shall choose δm so that in the $n = 0$ sector one eigenvalue always lies at zero energy. The Hamiltonian is then rewritten as.

$$(2.2)\qquad H = \delta m\,\bar{V}V + \mu\bar{\theta}\theta + g(V\bar{N}\bar{\theta} + N\bar{V}\theta) .$$

The eigenvectors of H in the n-th sector are

$$|\rangle = \alpha|1, 0, n\rangle + \beta|0, 1, n+1\rangle .$$

The eigenvalue equation $H|\rangle = E|\rangle$ leads to

$$(2.3)\quad [E-(n\mu+\delta m)]\alpha - g(n+1)^{\frac{1}{2}}\beta = 0 , \quad -g(n+1)^{\frac{1}{2}}\alpha + [E-(n+1)\mu]\beta = 0 .$$

Thus

$$(2.4)\qquad 2E = (2n+1)\mu + \delta m \pm \{(\mu-\delta m)^2 + 4g^2(n+1)\}^{\frac{1}{2}} .$$

By making $E=0$ a root for $n=0$, δm is determined uniquely:

$$(2.5)\qquad \delta m = g^2/\mu .$$

In this model μ is observable directly, so that the only parameter still to be determined is g. As in the conventional Lee model we choose g to reproduce the observations made in the lowest non-trivial sector. In our case this means that g is chosen to fit the level separation Δ for $n=0$, whence we find

$$(2.6)\qquad \Delta = \mu + g^2/\mu , \qquad g = \{\mu(\Delta-\mu)\}^{\frac{1}{2}} .$$

The inequality $\Delta > \mu$ is the analogue of the saturation condition $g < g_c$ in the conventional case. If observation yields $\Delta < \mu$ (which is just a departurs from the elementary rule that interacting levels repel each other), then g turns out imaginary and H non-Hermitian. For this contingency we make the convention

$$g = +i\{\mu(\mu-\Delta)\}^{\frac{1}{2}} .$$

Note that δm remains real as before.

Lastly we rewrite (2.5) and (2.6) in terms of the directly observed quantities:

$$(2.7)\qquad \delta m = \Delta - \mu ,$$

$$(2.8)\qquad E_{\pm} = (n\mu + \Delta/2) \pm \{\Delta^2/4 + n\mu(\Delta-\mu)\}^{\frac{1}{2}} .$$

Equation (2.8) exhibits the most characteristic feature of our model: for non-Hermitian H, *i.e.* negative $(\Delta-\mu)$, all the eigenvalues in all sectors with $n > n_c$ are complex, where

$$(2.9)\qquad n_c = \Delta^2/4\mu(\mu-\Delta) .$$

Being roots of a quadratic, the eigenvalues are of course complex conjugates of each other (*). The reason the eigenvalues in the $n = 0$ sector turn out real lies purely in our choice of δm. It is in the spirit of the investigation to adjust the two available constants in H to fit arbitrary data in this sector, and then ask what *kind*, if any, of observations in the higher sectors are compatible with this choice.

Next we turn to the determination of the eigenvectors. Denote by $|n_\pm\rangle$ the vectors in the n-th sector belonging to the eigenvalues with the $\pm$ signs in (2.8), and write

$$|n_\pm\rangle = \alpha_\pm\{|1, 0, n\rangle + (\beta/\alpha)_\pm|0, 1, n+1\rangle\}, \tag{2.10}$$

making the convention that α shall be real and positive (but see the Appendix for a different choice of phase). Equation (2.3) gives

$$(\beta/\alpha)_\pm \equiv \gamma_\pm = [E_\pm - (n\mu + \delta m)]/g(n+1)^{\frac{1}{2}},$$

$$\gamma_\pm = \frac{\{\mu - \Delta/2 \pm [\Delta^2/4 - n\mu(\mu-\Delta)]^{\frac{1}{2}}\}}{\{(n+1)\mu(\Delta-\mu)\}^{\frac{1}{2}}}. \tag{2.11}$$

In the Hermitian case $\gamma_\pm$ is real and the inner product of $|n_+\rangle$ with $|n_-\rangle$ of course vanishes.

If $\Delta < \mu$, however, the root in the denominator in (2.11) is imaginary; the root in the numerator is also imaginary for $n > n_c$. For $n < n_c$ we find

$$\langle n_+|n_-\rangle = 2\alpha_+\alpha_- \neq 0,$$

and also for $n > n_c$

$$\langle n_+|n_-\rangle = \frac{\alpha_+\alpha_-\{\mu - \Delta/2 - i[|\Delta^2/4 + n\mu(\Delta-\mu)|]^{\frac{1}{2}}\}^2}{\{1 + [(n+1)\mu(\mu-\Delta)]^{-1}\}} \neq 0.$$

2·2. *Heisenberg picture.* – The equations describing the model have been solved above in the Schrödinger picture by a method paralleling the usual treatment. It is possible to proceed differently, by solving the equations of motion for the field operators V, N, and θ, in the Heisenberg picture. Although this is a less economical way of arriving at the final result, we reproduce it in outline because it provides some insight into the circumstances that make the present version of the model completely soluble.

(*) We relegate to Appendix B the special case in which one sector is actually degenerate, *i.e.*, n_c is an integer. This would be analogous to the dipole ghost problem ([3]).

Writing down the Heisenberg equations of motion we find

$$\dot{\theta} = -i(\mu\theta + gV\bar{N}), \tag{2.14}$$

$$\dot{N} = -i\ V\bar{\theta}, \tag{2.15}$$

$$\dot{V} = -i(\delta m V + gN\theta). \tag{2.16}$$

We now recall that the operator $n = (\bar{V}V + \bar{\theta}\theta - 1)$ is a constant of the motion; this enables us to write the second time derivative

$$\ddot{V} = -i(\delta m + \mu)\dot{V} + [\mu\,\delta m - (n+1)g^2]V, \tag{2.17}$$

whence

$$V(t) = V_+ \exp[-iw_+(V)t] + V_- \exp[-iw_-(V)t]. \tag{2.18}$$

Here $V_\pm$ are time-independent operators,

$$V_\pm = \frac{\pm\{[w_\mp(V)_\mp\,\delta m]V(0) + gN(0)\theta(0)\}}{w_-(V) - w_+(V)}, \tag{2.19}$$

and the frequency operators are given by

$$2w_\pm(V) = (\delta m + \mu) \pm \{(\delta m - \mu)^2 + 4(n+1)g^2\}^{\frac{1}{2}}. \tag{2.20}$$

Similarly

$$\ddot{N} = -i(\mu - \delta m)\dot{N} - g^2(n+1)N, \tag{2.21}$$

whence

$$N(t) = N_+ \exp[-iw_+(N)t] + N_- \exp[-iw_-(N)t], \tag{2.22}$$

where

$$N_\pm = \pm\frac{\{w_\pm(N)N(0) + gV(0)\bar{\theta}(0)\}}{w_+(N) - w_-(N)}, \tag{2.23}$$

and

$$2w_\pm(N) = \delta m - \mu \pm \{(\delta m - \mu^2) + 4(n+1)g^2\}^{\frac{1}{2}}. \tag{2.24}$$

It is easy to verify that the eigenvalues of the frequencies $w_\pm(V)$ and $w_\pm(N)$ correspond to the energy differences between the pairs of states $|0, 0, n\rangle$ and $|n_\pm\rangle$, and $|0, 0, n+1\rangle$ and $|n_\pm\rangle$, respectively, these being the only pairs between which V and N have non-zero matrix elements.

By using the known expressions for $V(t)$ and $N(t)$ the equation of motion for θ can now be written as that for a forced harmonic oscillator. Alternatively, it is straightforward (though exceedingly tedious) to show that θ obeys a fifth order linear homogeneous equation with coefficient involving the constants of the motion n and $b = \bar{N}N + \bar{V}V$.

This linearity of the Heisenberg equations is perhaps the most striking illustration of the basic simplicity of the present model.

3. – Indefinite metric.

If $\Lambda < \mu$, clearly nothing can be done to remedy the unfortunate fact that H has complex eigenvalues. However, one can ensure that its expectation values are real by introducing an indefinite metric, *i.e.* a metric operator η; in the usual manner [2,4]. The norm of state $|n\rangle$ is then redefined as

$$\langle |\eta| \rangle \tag{3.1}$$

and the expectation value of an operator Q in this state as

$$\langle |\eta Q| \rangle , \tag{3.2}$$

(3.1) being a special case of (3.2) for $Q = 1$.

To achieve its purpose η is chosen to make H self-adjoint in the sense

$$\eta H - \bar{H}\eta = 0 ; \tag{3.3}$$

this can be achieved by

$$\eta = (-1)^{\bar{V}V} = 1 - 2\bar{V}V ; \tag{3.4}$$

($n = (-1)^{\bar{N}N}$ would serve equally well). Note that η is Hermitian.

The following relations are found to hold when $n < n_c$, *i.e.* for real eigenvalues:

$$\langle n_+ | \eta | n_- \rangle = 0 , \tag{3.5}$$

$$\langle n_+ | \eta | n_+ \rangle > 0 , \tag{3.6}$$

$$\langle n_- | \eta | n_- \rangle < 0 . \tag{3.7}$$

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[4] L. K. Pandit: *Suppl. Nuovo Cimento*, **11**, 157 (1959).

For $n > n_c$, *i.e.* the conjugate complex eigenvalues, we have

$$\langle n_+ | \eta | n_- \rangle \neq 0 , \tag{3.8}$$

$$\langle n_+ | \eta | n_+ \rangle = 0 , \tag{3.9}$$

$$\langle n_- | \eta | n_- \rangle = 0 . \tag{3.10}$$

States belonging to different sectors are, of course, orthogonal as before.

The orthogonality relations (3.5), (3.9), and (3.10) are consequences of standard theory ([4]) and can also easily be verified explicitly; (3.6) is characteristic of the model, and (3.7) follows immediately from (3.6) and (3.5). To establish (3.6) we note

$$\langle n_\pm | \eta | n_\pm \rangle = \alpha_\pm^2 \{ -1 + |\gamma_\pm|^2 \} , \tag{3.11}$$

where from (2.11)

$$\gamma_+ = \frac{\mu - \Delta/2 + \{\Delta^2/4 - n\mu(\mu - \Delta)\}^{\frac{1}{2}}}{i\{(n+1)\mu(\mu - \Delta)\}^{\frac{1}{2}}} . \tag{3.12}$$

Now $|\gamma_+|^2 \gtrless 1$ according to

$$\Delta^2/2 + 2(\mu - \Delta/2)\{\Delta^2/4 - n\mu(\mu - \Delta)\}^{\frac{1}{2}} \gtrless 2n\mu(\mu - \Delta) . \tag{3.13}$$

But the right hand side is less than $2n_c\mu(\mu - \Delta) = \Delta^2/2$ while the left side is always greater than $\Delta^2/2$ so that indeed $|\gamma_+|^2 > 1$.

It is convenient at this point to renormalize the states so that for $n < n_c$

$$\langle n_\pm | \eta | n_\pm \rangle = \pm 1 . \tag{3.14}$$

4. – Discussion.

The norm of a state

$$| \quad \rangle = \sum_{n,i} c_i(n) | n_i \rangle ,$$

where $i = \pm$ is

$$\langle | \eta | \rangle = \sum_{n < n_c} \{ |c_+(n)|^2 - |c_-(n)|^2 \} + \sum_{n > n_c} 2 \operatorname{Re} \bar{c}_+(n) c_-(n) \langle n_+ | \eta | n_- \rangle .$$

Difficulties of interpretation due to formally negative probabilities can certainly be avoided by imposing the subsidiary condition that physically admissible state vectors when expanded in terms of energy eigenstates shall contain no components with negative or zero norm. The orthodox quantum-mechanical theory of measurement can then be taken over, restricting possible results of energy measurements to the real n_- eigenvalues. However the Hilbert space spanned by these admissible state vectors is seen to have shrunk to finite dimensions, and the expectation value of the Hamiltonian is always less than the greatest permissible eigenvalue.

Next, we try to remove this limitation by admitting more vectors, and ask whether a consistent theory of measurement still exists in this extended space. Clearly negative norm states cannot be admitted if the statistical interpretation is to be maintained. Hence we try to relax the condition excluding vectors with zero norm, *i.e.* belonging to complex eigenvalues. If at most one member of each pair is admitted ([5]) neither the norm nor the energy expectation values are affected, since the new term is orthogonal to all terms (including itself) in the wavefunction. The amplitude of such admixtures will grow or decay exponentially with time.

Suppose next that we admit both members in some normalized combination, $|n\rangle$. This differs qualitatively from the vectors already allowed by not being an energy eigenstate:

$$|n\rangle = \alpha\{|n_+\rangle + \gamma|n_-\rangle\}\,.$$

Since $|n_+\rangle$ and $|n_-\rangle$ separately have zero norm they are, so far, undefined to within an arbitrary factor. Because only their inner product is relevant, we can, and do, choose these factors so that

$$\langle n_+|\eta|n_-\rangle = +1\,. \tag{4.1}$$

Then we have

$$\langle n|\eta|n\rangle = \alpha^2(\gamma + \gamma^*) = 1\,, \qquad \alpha^{-2} = 2\,\mathrm{Re}\,\gamma\,.$$

Further,

$$\langle n|\eta H|n\rangle = 2\alpha^2\,\mathrm{Re}\,(E_-\gamma)\,.$$

In order to specify the newly admitted states as closely as possible we stipulate further that γ be real and positive; since its magnitude varies with

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([5]) W. Pauli: in *Proceedings of the 1958 Annual International Conference on High-Energy Physics at CERN* (CERN, Geneva, 1958).

time no more stringent conditions can be imposed. Thus

$$\langle n|\eta H|n\rangle = \mathrm{Re}\, E_- = n\mu + \Delta/2\,, \tag{4.2}$$

and the limitation on the expectation value of H is indeed lifted.

However, if our theory is to be acceptable it must contain some sensible statements about the possible results of energy measurements performed on the system when in a state described by $|n\rangle$. Since $|n\rangle$ is not an energy eigenstate one would not expect that all such measurements yield a given sharp value. But the theory must be capable of specifying the statistical distribution of such results. Actually, however, it fails to do this, as can be seen already by computing explicitly the second moment of this distribution, *i.e.* the mean square fluctuation δE^2:

$$\delta E^2 = \langle n|\eta H^2|n\rangle - (\langle n|\eta H|n\rangle)^2 = -(\mathrm{Im}\, E)^2\,. \tag{4.3}$$

The result $\delta E^2 < 0$ is independent of the restrictions placed on $|n\rangle$. In the general case of complex γ one finds

$$\delta E^2 = -(\mathrm{Im}\, E)^2 |\gamma|^2/(\mathrm{Re}\,\gamma)^2\,.$$

We are forced to the conclusion that no consistent and sensible theory of measurement is available for this particular model, and that we have failed in our attempt to extend the space of physical vectors in the desired manner.

* * *

It is a pleasure to take this opportunity to acknowledge many basic and exhaustive discussions with Dr. G. Sandri, to whom is due the original observation that the above model is completely soluble; and to thank Professor J. R. Oppenheimer for extending to me the support and hospitality of the Institute for Advanced Study.

Appendix A

Renormalization.

It is of some interest to compare the treatment given in the text with the following conventional renormalization procedure. We define in the usual way

$$g_R = Z^{\frac{1}{2}} g\,, \qquad V_R = Z^{-\frac{1}{2}} V\,,$$

where $Z^{\frac{1}{2}}$ is the coefficient of $|1, 0, 0\rangle$ in that state of the $n = 0$ sector which goes adiabatically into $|1, 0, 0\rangle$ as the coupling is switched off.

The renormalized form of the Hamiltonian given in (1.2) is

$$H = H_R = \delta m Z \overline{V}_R V_R + \mu \bar{\theta}\theta + g_R(V_R \bar{N}\bar{\theta} + N \overline{V}_R \theta) ,$$

and the commutation rule for V becomes

$$[V_R, \overline{V}_R] = Z^{-1} .$$

We find immediately

$$Z = 1 - g_R^2/\mu^2 = \mu/\Delta$$

and therefore

$$g_R^2 = g^2\mu^2/(g^2 + \mu^2) , \qquad g_{\text{crit}} = \mu .$$

For $g_R > g_c$, g is imaginary.

We can express any result of the theory in terms of the renormalized quantities only. In particular,

$$\Delta = \mu^3/(\mu^2 - g_R^2) ,$$

so that

$$g_R^2 = \mu^2(\Delta - \mu)/\Delta .$$

Note that all expressions can be expanded in powers of either g_R^2 or g_R^{-2} in contrast to the situation in the full Lee model.

There are three possible outcomes of an experimental determination of Δ with different implications for the reality of the coupling constants:

(I) $-\infty < \Delta < 0$: g imaginary, g_R real, $Z^{\frac{1}{2}}$ imaginary;

(II) $0 < \Delta < \mu$: g imaginary, g_R imaginary, $Z^{\frac{1}{2}}$ real;

(III) $\mu < \Delta < \infty$: g real, g_R real, $Z^{\frac{1}{2}}$ real.

Case (II), with g_R imaginary although it has no counterpart in the conventional model is no more pathological than the other non-Hermitian contingency.

If we adopt the indefinite metric in this approach, the norms of the energy eigenstates turn out differently than before. The norm of the state

$$Z^{\frac{1}{2}}|1, 0, 0\rangle - (1 - Z)^{\frac{1}{2}}|0, 1, 1\rangle$$

is $+1$ in cases (I) and (III) and -1 in case (II), while the norm of the other eigenstate in the $n = 0$ sector has the opposite sign.

This difference stems from the convention regarding the overall phase of the vectors. In the present treatment this phase is defined so that the coefficient of $|1, 0, 0\rangle$ is adiabatically continuous from the Hermitian case, whereas in the text we took it to be always real and positive.

Finally, it may be worth-while to mention the analogy between the two vectors in the $n = 0$ sector and the corresponding ones in the full Lee model. One of our two vectors of course corresponds to the normal physical V-particle. In the Hermitean case, the other vector is just the «anomalous scattering state» of KÄLLÉN and PAULI [2]: it recedes in energy to $+\infty$ as g_R^2 approaches g_{crit} from below; in the non-Hermitian case it is this state that reappears at $-\infty$ as the negative-norm ghost state which is not therefore in any sense a «new» one.

APPENDIX B

Dipole ghost.

A dipole ghost arises in the n-th sector if Δ and μ are such that

$$n = \Delta^2/4\mu(\mu - \Delta) .$$

In that case one finds

$$\gamma' = -i .$$

Thus H has only *one* eigenvector in this sector, namely

$$|n_-\rangle = \alpha\{|1, 0, n\rangle - i|0, 1, n+1\rangle\} .$$

The combination orthogonal to this is

$$|n_+\rangle = \alpha\{|1, 0, n\rangle + i|0, 1, n+1\rangle\} .$$

The condition that $|n \pm\rangle$ should be eigenvectors is from

$$H|n \pm\rangle = \alpha\{[\delta m + n\mu \pm ig\sqrt{n+1}]|1, 0, n\rangle + \\ + [(n+1)\mu \mp ig\sqrt{n+1}]i|0, 1, n+1\rangle\}$$

easily seen to be

$$(\delta m - \mu) \pm 2ig\sqrt{n+1} = 0 ,$$

which of course cannot both be fulfilled simultaneously, although the difference between them is purely one of convention, *i.e.* of the definition of the phase of g when it is complex; (cf. the equation following (2.6)).

The norms of $|n \pm\rangle$ with respect to η are immediately seen to be zero.

RIASSUNTO (*)

Consideriamo un modello di Lee con un solo oscillatore del campo bosonico. Le equazioni di Heisenberg risultano lineari e il problema è solubile in tutti i settori. Lo spettro è interamente ed essenzialmente discreto, e le questioni di interesse fisico non si riferiscono allo scattering ma alle misure dirette dell'energia. In particolare si possono esplicitamente individuare tutte le conseguenze della introduzione di una metrica indefinita; risulta che non si può dare nessuna coerente interpretazione fisica.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

The Kemmer β-Formalism for Particles of Spin One-Half (*).

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(ricevuto il 23 Maggio 1960)

Summary. — A linearization of the two-component second order spinor equation of Feynman and Gell-Mann for spin one-half particles leads to a ten component equation of the Kemmer type. Up to now such equations were only known for integral spin particles.

1. – Introduction.

In the conventional field theories particles with integral or half-integral spin values are treated by quite different formalisms. The field quantities are tensors for the former and spinors for the latter case. In the attempts to unify and to classify the elementary particle-interactions it may be desirable to describe all free particles by the same formalism. The difference will appear in the interaction terms. This objective can be accomplished by passing to some unusual representations. One such possibility has been described before [1] where a theory of spin one half particles is developed as an exact counterpart of a theory for particles with spin 0. In this paper we discuss another possibility by showing that the well-known Duffin-Kemmer β-formalism of spin 0 and 1 particles [2] can also be extended to the spin one half case, if the interaction term is suitably modified. The spin value will be determined by the

(*) Supported in part by the Air Force Office of Scientific Research.

(**) Now at Osmania University, Hyderabad, India.

(1) A. O. Barut: *Ann. of Phys.*, **5**, 95 (1958).

(2) R. J. Duffin: *Phys. Rev.*, **54**, 1114 (1938); N. Kemmer: *Proc. Roy. Soc.*, A **173**, 91 (1939).

interaction term. These Kemmer-type equations for the spin one half case follow directly from the two-component second order equation of Feynman and Gell-Mann ([3]). It is important to remark that although there is a one-to-one correspondence between the solutions of the Dirac equation and that of the Feynman–Gell-Mann equation, these two equations have different invariance properties. For example, the latter is only PC invariant, whereas the Dirac equation conserves separately P and C, if $\boldsymbol{\sigma}$ in the Feynman–Gell-Mann equation are interpreted as spin operators.

2. – Duffin-Kemmer theory.

The free particle Duffin-Kemmer equation ([2]) is

$$(\beta^\mu p_\mu + m)\psi = 0\,, \tag{1}$$

where β^μ satisfy the following commutation relations

$$\beta^\mu\beta^\nu\beta^\sigma + \beta^\sigma\beta^\nu\beta^\mu = \beta^\mu g^{\nu\sigma} + \beta^\sigma g^{\nu\mu} \tag{2}$$

and no further specification of the β^μ is needed.

There are 126 linearly independent quantities in the Kemmer algebra generated by the four β-matrices satisfying (2). The algebra is semi-simple and has three irreducible representations of dimensionalities 1, 5 and 10.

The equations (1) and (2) give us also representations (reducible) of the Lorentz group of dimensions 5 and 10. The invariance of (1) under Lorentz transformations $x' = Lx$ with $\psi'(x') = D(L)\psi(x)$ implies

$$D^{-1}\beta^\mu D = L^\mu_\nu \beta^\nu\,. \tag{3}$$

The D-matrices are reducible. For 5×5 matrices $D = D^{\frac{1}{2}\frac{1}{2}} + D^{00}$, for 10×10 matrices $D = D^{10} + D^{01} + D^{\frac{1}{2}\frac{1}{2}}$, where $D^{ij'}$ are the usual representations of the homogeneous Lorentz group (or of C_2, the unimodular group). There are 126 covariant expressions bilinear in the ψ and $\overline{\psi} = \psi^+\eta$ with $\eta = 2\beta^{0^2} - 1$. The covariant spin operators are given by $-i[\beta^\mu, \beta^\nu]$. $S_3 = -i[\beta^1, \beta^2]$ has eigenvalues 0, 0, 0, $+1$, -1 for the 5×5 case, twice as many for the 10×10 case. However, the expectation value of the spin has the correct value of 0 and 1, respectively.

([3]) R. P. Feynman and M. Gell-Mann: *Phys. Rev.*, **109**, 193 (1958).

For electromagnetic interactions p_μ in eq. (1) will be replaced by $P_\mu = p_\mu - eA_\mu$.

3. – Linearization of the Feynmann–Gell-Mann equation (*).

We start from the two component second order Feynman–Gell-Mann equation ([1,3])

$$\left(P^2 - m^2 + \frac{e}{2}\sigma^{\mu\nu}F_{\mu\nu}\right)\varphi = 0 \,. \tag{4}$$

Note that this equation is different from the four-component, second order Dirac equation $(P^2 - m^2 + (e/2)\gamma^{\mu\nu}F_{\mu\nu})\psi = 0$ from which it can be derived.

$$\gamma^{\mu\nu} = \frac{1}{2i}[\gamma^\mu, \gamma^\nu], \quad \sigma^{\mu\nu} = \frac{1}{i}\begin{pmatrix} 0 & \sigma_1 & \sigma_2 & \sigma_2 \\ & 0 & -i\sigma_3 & i\sigma_2 \\ & & 0 & -i\sigma_1 \\ & & & 0 \end{pmatrix} = -\sigma^{\nu\mu} \,. \tag{5}$$

Let us introduce four new two-component quantities φ^μ defined by

$$P^\mu\varphi = (p^\mu - eA^\mu)\varphi = m\varphi^\mu \,. \tag{6}$$

Then φ and φ^μ, together, define a 10-component quantity. (6) inserted in (4) gives

$$P_\mu\varphi^\mu - m\varphi + \frac{e}{2m}\sigma^{\mu\nu}F_{\mu\nu}\varphi = 0 \,. \tag{7}$$

In the absence of electromagnetic interactions equations (6) and (7) can be written in the Kemmer form

$$(\beta^\mu p_\mu + m)\Psi = 0 \,, \qquad \Psi = \begin{pmatrix} \varphi^\mu \\ \varphi \end{pmatrix}, \tag{8}$$

where the β matrices are 10×10, same as for spin 0 and 1 case (eq. (1), (2)) with the matrix elements 1 replaced by 2×2 I-matrices. The additional term in (7) gives an additional term in (8) so that the final equation with the

(*) $c = \hbar = 1$, $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$, $g^{\mu\nu} = 0$ for $\mu \neq \nu$, $A^\mu = (A^0, \boldsymbol{A})$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

electromagnetic interactions is

$$\left(\beta^\mu P_\mu + m - \frac{e}{2m}\Lambda\right)\Psi = 0\,, \tag{9}$$

where

$$\Lambda = \begin{pmatrix} \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \sigma^{\mu\nu}F_{\mu\nu} \end{pmatrix}$$

the dots being 2×2 zero matrices.

We remark that if we had started with the second order Dirac equation we would also arrive at an equation of the form (9), but with $\sigma^{\mu\nu}$ replaced by $\gamma^{\mu\nu}$. However, Ψ would then have 20 components instead of 10.

The effect of the electromagnetic field in eq. (9) is two-fold: one is in $P_\mu = p_\mu - eA_\mu$, as for the spin 0 case, the other in the additional term $-(e/2m)\Lambda$ which is characteristic for spin one-half particles.

The free particle part of (9) is reducible in two 5 component equations. The « bare » particle has, therefore, a spin value zero. The observed spin value is due to the term $-(e/2m)\Lambda\Psi$ in eq. (9). This fact is also true for the original second order equation (4), where the free field part of the equation reduces into two Klein-Gordon equations. One would expect therefore that the usual second quantization procedures have to be modified for the free fields, if one wants to quantize a two-component Klein-Gordon equation with anticommutators and to obtain a positive definite energy. These questions will be discussed separately.

Eq. (9) can be put in the Hamiltonian form with

$$H = eA_0 + (p_k - eA_k)[\beta^k, \beta^0] - m\beta^0 + \frac{ei}{2m}F_{\sigma\mu}(\beta^\mu\beta^0\beta^\sigma + g^{0\sigma}\beta^\mu) + \tag{10}$$

$$+ \frac{e}{2m}\beta^0\Lambda - \frac{e}{2m^2}(p_k - eA_k)\beta^k\beta^0\Lambda\,,$$

k is summed over 1, 2, 3.

For $\Lambda = 0$ we get back the Hamiltonian form of KEMMER [2]. The derivation of eq. (10) is similar to the Kemmer case except that one has to make use of the relation $(1-\beta^{0^2})\Lambda = 0$ in order to separate the Hamiltonian.

The removal of the six redundant component of eq. (9) can be done by Heitler's method (4) exactly analogous to the spin 0 case and will not be repeated here.

Since the Maxwell equations can also be written in the form (5) $(\beta^{\mu}p_{\mu}+\gamma)\Psi=0$, where $\gamma^2=\gamma$ and $\{\gamma,\beta^{\mu}\}=\beta^{\mu}$, we conclude that in the β-formalism we have a possibility of uniformly describing all particles (6).

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(4) W. HEITLER: *Proc. Roy. Irish Acad.*, **49**, 1 (1943).

(5) HARISH-CHANDRA: *Proc. Roy. Soc.*, A **186**, 502 (1946).

(6) Form $m=0$ eq. (9) breaks down. However, starting from (4) with $m=0$ we can, instead of eq. (6), formally define $P^{\mu}\varphi=\varphi$ and obtain an eq. similar to (9).

RIASSUNTO (*)

Una linearizzazione della equazione spinoriale del secondo ordine a due componenti di Feynman e Gell-Mann per particelle di spin $\frac{1}{2}$ porta ad una equazione a dieci componenti del tipo di Kemmer. Sinora queste equazioni erano note solo per particelle a spin intero.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

Nuclear Hyperfine Structure of Hydrazylic Free Radicals (*).

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(ricevuto il 27 Maggio 1960)

Summary. — To obtain information on the electronic structure of hydrazylic free radicals, the electron spin resonance hyperfine structure of a group of such radicals derived from 1-1-diphenyl-2-benzoyl-hydrazyl has been investigated as a function of suitable substitutions of atoms or groups at different positions in the benzoylic ring. It has been found that the introduction of a nitric group at an « ortho » position to the carbonyl group causes a change of the spectrum which, on the contrary, is not observed if the same group is introduced at a « para » position. This indicates a considerably larger $I \cdot S$ interaction in the former case, as compared with the latter, namely a charge transfer preferentially centered on the ortho positions.

1. – Introduction.

As is well known, the investigation of the electron spin resonance (E.S.R.) spectrum of free radicals makes possible the determination of the parameters of the spin Hamiltonian of their unpaired electron and, in particular, the determination of the most effective terms of the $I \cdot S$ coupling. This allows a theoretical analysis of the electronic structure of the systems to be performed and/or an evaluation of the charge transfer to be made.

Earlier and more recent work [1-4] on E.S.R. in diluted solutions of DPPH

(*) This work forms part of a program of research undertaken with the joint support of the Sicilian Comitato Regionale Ricerche Nucleari and the Consiglio Nazionale delle Ricerche, for which the authors wish to express their sincere gratitude.

(1) C. A. Hutchison, R. C. Pastor and A. G. Kowalsky: *Journ. Chem. Phys.*, **20**, 534 (1952).

(2) H. S. Jarrett: *Journ. Chem. Phys.*, **21**, 761 (1953).

(3) C. Kikuchi and V. W. Cohen: *Phys. Rev.*, **93**, 394 (1954).

(4) R. G. Bennett and A. Henglein: *Journ. Chem. Phys.*, **30**, 1117 (1959).

and derivatives have shown the $I \cdot S$ interaction of the unpaired electron with the two hydrazylic nitrogen nuclei. A similar interaction with the remaining nitrogen nuclei had not been evidenced, although a considerable charge transfer is suggested by their remarkable stability ([5]).

In the present work such a charge transfer in hydrazylic radicals has been studied and evidenced by the E.S.R. technique and by suitable substitutions of one or more hydrogen nuclei.

2. – Experimental methods and results.

2·1. *Techniques.* – The free radicals were prepared, following the Goldschmidt method ([6]), starting from 1-1-diphenyl-hydrazine and acid chlorides in ether and by oxidizing the hydrazine thus obtained with potassium ferricyanide.

The E.S.R. spectrograph used has already been described ([7]) and here we will only recall the features which are of interest for the present work. The magnetic field is automatically controlled and swept and has a relative stability of about 10^{-5} and an even better relative homogeneity (over the sample). In our experiments the field was low-frequency modulated with an amplitude of about 1 gauss and the derivative of the absorption lines was recorded. The microwave frequency (of about 9300 MHz) was directly locked by A.F.C. to the cylindric TE_{011} reflection cavity. The cavity arm was protected by a pyrex tube and immersed in a dewar containing solid carbon dioxide. A thermocouple and a system of Variac—feeded resistors allowed stabilization and control of the temperature anywhere between about 0 °C and — 78.5 °C.

The E.S.R. spectra were recorded using carbon tetrachloride solutions at initial concentrations of the order of 300 mg per liter. Due to the fact that the free radicals were in equilibrium with their non-magnetic dymers in the solution, the maximum actual concentration of the radicals was estimated to be about 10^{-4} mol/l'ter or less.

The solutions were prepared at about — 20 °C in a nitrogen atmosphere and thereafter were put in small pyrex ampoules inside the precooled cavity. Care was taken that the solutions could never reach higher temperatures or be surrounded by gases other than nitrogen.

2·2. *Results.* – The free radicals which have been investigated are shown in Table I together with the main experimental results. They were obtained

([5]) L. Pauling and G. W. Wheland: *Journ. Chem. Phys.*, **1**, 362 (1933).

([6]) S. Goldschmidt: *Ann. der Chemie*, **437**, 124 (1924).

([7]) M. B. Palma-Vittorelli and M. U. Palma: *Suppl. Nuovo Cimento*, **7**, 139 (1958).

from the radical:

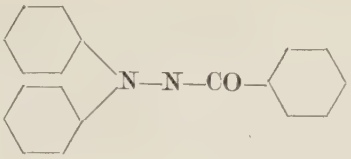
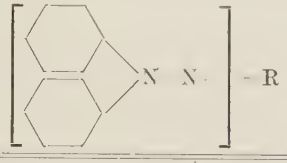


TABLE I. – *E.S.R. absorption of a series of free radicals in carbon tetrachloride. Concentration* $< 10^{-4}$ mol/litre. *Temperature about* -20 °C.



No.	Radical	Number of resolved absorption lines	Distance between the principal lines (gauss)
1	R—CO–C$_6$H$_5$ [8]	5	8.2
2	R—CO–C$_6$H$_4$–OCH$_3$	5	8.2
3	R—CO–C$_6$H$_4$–Cl	5	8.3
4	R—CO–C$_6$H$_4$(Cl)$_2$	5	8.1
5	R—CO–C$_6$H$_4$–NO$_2$	5	8.1
6	R—CO–C$_6$H$_3$(NO$_2$)$_2$	5×3	7.8
7	R—CO–C$_6$H$_3$(NO$_2$)$_2$	5×3	7.7
8	R–C$_6$H$_2$(NO$_2$)$_3$ (DPPH)	5	8

([8]) Preliminary data have already been given for this radical: I. S. CICCARELLO, T. GAROFANO and M. SANTANGELO: *Nuovo Cimento*, **12**, 389 (1959).

by substitution of one or more hydrogens of the benzoylic ring with different atoms or groups. In the following, the different radicals will be labeled by numbers in accordance with Table I.

At a temperature of — 20 °C, the spectra of all the radicals of Table I exhibit an easily detectable hyperfine structure of 5 main lines which extends in a range of about 40 gauss. When the temperature is slowly lowered down to — 78.5 °C, the different hyperfine structure lines are fused in only one almost symmetrical line whose width, in the conditions of our experiments, is about 15 gauss. This effect is probably due to a phase-separation between radicals and solvent, following the freezing, and to the subsequently enhanced narrowing exchange interaction (9). The fading of the structure is in fact not complete if the freezing occurs very rapidly.

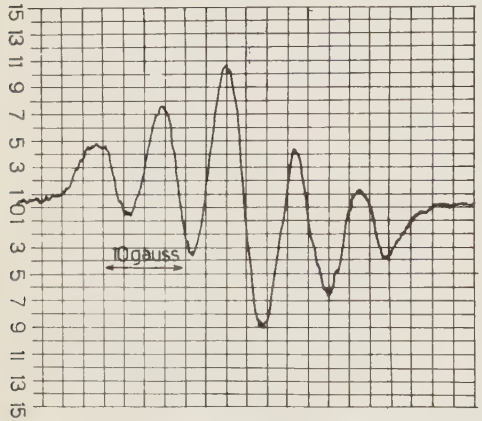


Fig. 1. – E.S.R. recorded spectrum (derivative) of radical No. 3. Radicals No. 1, 2, 4 and 5 exhibit a very similar spectrum.

The optimum temperature for both signal and resolution is about — 20 °C and the reported data refer to this temperature. A further increase of the temperature causes a decrease of the signal, in agreement with the expected decomposition of the radicals.

The E.S.R. spectrum (derivative) of radical No. 3 is shown in Fig. 1. Five equally spaced h.f.s. lines are present, with an intensity ratio close to 1 : 2 : 3 : 2 : 1. A very similar behaviour is exhibited by radicals No. 1, 2, 4, 5 with an almost constant width of the hyperfine structure lines (about 5 gauss each). The g-values are very close to that of the DPPH. Further data concerning these radicals are found in Table I.

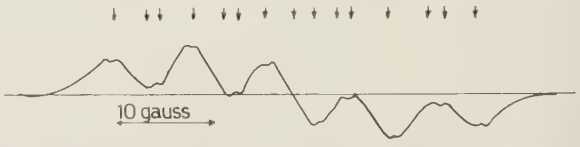


Fig. 2. – E.S.R. spectrum tracing (derivative) of radical No. 7. Radical No. 6 exhibits a very similar spectrum.

In Fig. 2 is shown a tracing of the E.S.R. (derivative) spectrum of radical No. 7 in which are visible appreciable modifications, compared with the spectrum of Fig. 1. The five principal lines show now a further, barely detectable, structure of 3 lines each (a total of 15 lines), which are anyhow con-

(9) A. Van Roggen, L. Van Roggen and W. Gordy: *Phys. Rev.*, **105**, 50 (1957).

stantly reproducible. Radical No. 6 shows a spectrum similar to that of radical No. 7. In both cases the g-values are again very close to that of the DPPH.

The No. 8 radical exhibits a spectrum similar to that of Fig. 1, but rather less resolved and with no further structure.

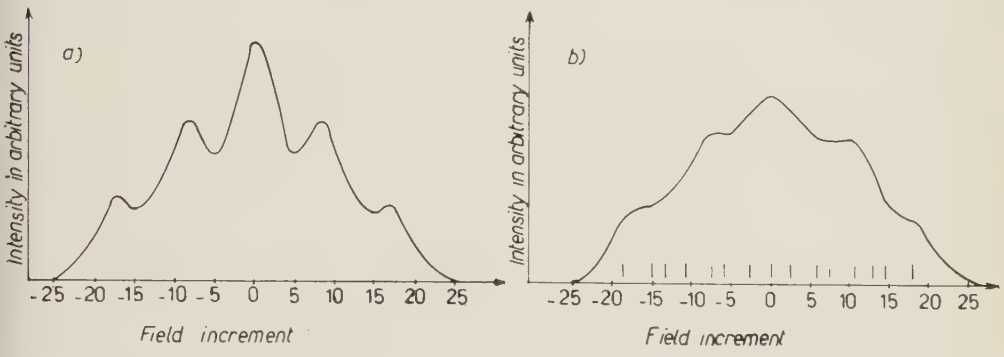


Fig. 3. – *a*) E.S.R. of radical No. 3 numerically integrated from the recorded (derivative) spectrum. *b*) E.S.R. of radical No. 7 numerically integrated from the recorded (derivative) spectrum. Field increments are given in gauss. In both cases *a*) and *b*) the concentrations are such to give optimum resolutions.

Fig. 3*a* shows a numerically integrated spectrum of the type depicted in Fig. 1. A comparison with Fig. 3*b* (which shows the integrated spectrum of Fig. 2, together with the position of the barely resolved lines), illustrates the difference in width and therefore in resolution of the main lines in the two cases.

3. – Discussion of the experimental results.

The differences between the types of spectra of Figs. 1 and 2 or 3*a* and 3*b* are the following: *a*) the presence of 15 h.f.s. lines in the latter, compared with the 5 lines of the first; *b*) the remarkably better resolution obtained in the case of Figs. 1 and 3*a* compared with Figs. 2 and 3*b*.

The type of spectrum shown in Fig. 1 is described by the following spin Hamiltonian (with the well known meaning of the symbols):

$$\mathcal{H} = g\beta H \cdot S + a_1 I_{N_1} \cdot S + a_2 I_{N_2} \cdot S ,$$

where N_1 and N_2 stand for the two central (hydrazynic) nitrogen nuclei and a_1, a_2 for the related coupling constants. The five lines experimentally found, with the found intensity ratios are accounted for by taking $a_1 = a_2$ (*i.e.* assuming the equivalence of the two nitrogen nuclei). For the general case $a_1 \neq a_2$,

the behaviour of the spectrum as a function of a_1/a_2 has been given by C. KIKUCHI and V. W. COHEN [3].

The type of spectrum of Fig. 3 may be described by taking into account, beside the mentioned interaction with the central nitrogen nuclei, a smaller interaction with the nitrogen nucleus in an « ortho » position to the carbonyl. In other words, the new type of spectrum is described by the following spin Hamiltonian:

$$\mathscr{H} = g\beta H \cdot S + a_1 I_{N_1} \cdot S + a_2 I_{N_2} \cdot S + a_3 I_{N_3} \cdot S ,$$

where the subscript 3 refers to the « ortho » position nitrogen. By assuming again $a_1 = a_2$ and $a_1 = a_2 \gg a_3$ one finds the 15 experimentally recorded lines.

The separation between the main lines in the investigated radicals varies between 8.2 and 7.7 gauss; the slight progressive variation may be explained in terms of a corresponding progressive increase of the charge transfer on the benzoylic ring.

The a_1 and a_2 values are easily deduced from Table I; the order of magnitude of the a_3 value may be estimated from Fig. 2.

From the foregoing considerations and by taking into account the fact that radical No. 5 gives an E.S.R. spectrum of the type shown in Fig. 1 whereas radical No. 7, which differs only in the position of the nitric group, exhibits a spectrum of the type shown in Fig. 2, one may conclude that the present experiments specify the interaction between the unpaired electron and the nuclei in ortho and para positions and give evidence for a sizable interaction only with the nuclei at ortho positions. It seems reasonable to think that this interaction has not been revealed in the DPPH and in other hydrazylic radicals probably for the reason that in this case it gives rise to many lines which remain unresolved. This is in agreement with the worst resolution of the spectrum of the DPPH, compared with that of radicals 1, 2, 3, 4, 5, in which the mentioned interaction is not present.

* * *

We wish to acknowledge several stimulating friendly discussions with M. B. PALMA-VITTORELLI and M. U. PALMA.

RIASSUNTO

Allo scopo di ottenere informazioni sulla configurazione elettronica di radicali idrazilici liberi, sono state studiate le strutture iperfini di E.S.R. di un gruppo di radicali derivanti dall'1,1-difenil-2-benzoil-idrazile, per opportune sostituzioni di atomi o gruppi in differenti posizioni dell'anello benzoilico. È stato osservato che l'introduzione di un gruppo nitrico in posizione orto al carbonile provoca un mutamento dello spettro che, al contrario, non è osservato se lo stesso gruppo è introdotto in posizione para. Ciò fa presupporre nel primo caso una maggiore interazione $I \cdot S$ e cioè un maggiore trasporto di carica sull'azoto nitrico in posizione orto.

Sign of the Nuclear Potential for the K^+-Meson.

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(ricevuto il 30 Maggio 1960)

Summary. — The sign (*) of the nuclear potential for the K^+-meson is determined from the small angle distribution of the elastic scattering in G-5 nuclear emulsion. The investigation has been carried out in the low kinetic energy region (< 60 MeV). The Born approximation has been applied to analyse the data and it is found that the nuclear potential is repulsive.

1. – Introduction.

For this investigation we used two stacks of G-5 nuclear emulsion which had been irradiated with K^+-mesons from the Bevatron at Berkeley [1]. In total 210 K^+-meson tracks were traced in the low kinetic energy region in search for scattering events the projected angles of which were $\geqslant 2°$. The analysis was restricted to the data of the elastic collisions of the K^+-mesons with nuclei of the emulsion other than hydrogen. The experimental cross-sections have been corrected for the geometric loss due to the unfavourable orientation of the scattering event [2,3].

In this work the data of the elastic scattering of low energy K^+-mesons have been analysed with the help of the Born approximation [4-6]. Instead

(*) The determination of the nuclear potential value in the energy region of (40÷80) MeV will be reported in a separate paper.

[1] R. W. Birge, R. P. Haddock, L. T. Kerth, J. R. Peterson, J. Sandweis, D. H. Stork and M. N. Whitehead: *Phys. Rev.*, **99**, 329 (1955).

[2] T. G. Lim and S. J. Bosgra: *Nuovo Cimento*, **8**, 340 (1958).

[3] T. G. Lim: *Thesis* (Amsterdam).

[4] M. E. Rose: *Phys. Rev.*, **73**, 279 (1948).

[5] A. Pevsner and J. Rainwater: *Phys. Rev.*, **100**, 1431 (1955).

[6] T. F. Hoang, M. F. Kaplon and R. Cester: *Phys. Rev.*, **107**, 1698 (1957).

of the method adopted by OSBORNE (7) and DAVIS (8) who considered the detailed mechanism of the K^+-meson-nucleon scattering, we compute the elastic differential cross-sections in terms of the nuclear potential V, which we take real. We restrict ourselves to the determination of the sign of V, as for the low energy K^+-meson, the method is less appropriate to estimate the value of the nuclear potential (see HOANG *et al.* (6)).

2. – Elastic and inelastic events.

For the determination of the relative kinetic energy loss, wherever possible about 1000 grains were counted on the K^+-meson track, before and after each scattering. Of the pellicle, only the region between ~ 40 μm from the air side and ~ 40 μm from the glass side were used for counting the grains. Grain counts were usually carried out for the kinetic energies > 30 MeV, as it becomes very difficult for energies that are too low.

To find a criterion for classifying the scattering events into elastic and inelastic ones, a histogram was made of the measured relative kinetic energy changes at the scattering centre (Fig. 1). Attributing positive values of the distribution to errors in counting the number of grains and to statistical fluctuation, these events are taken to be elastic ones. An approximation of the relevant error function is obtained by reflecting the positive part of the distribution with respect to the vertical axis through the origin. The standard deviation is found to be 0.05. All scattering events with relative kinetic energy loss of 10% and less are classified as elastic ones, and scattering events with relative energy loss $> 10\%$ as inelastic ones.

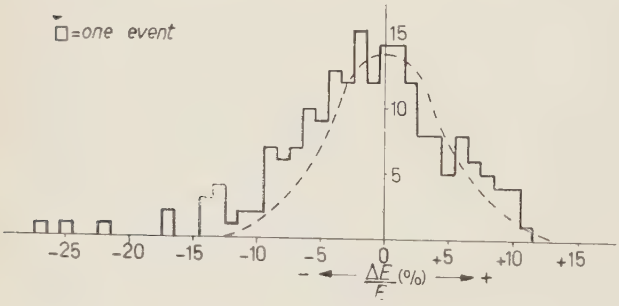

Fig. 1. – Histogram of relative energy changes.

The mean free path for the inelastic scattering in the energy region of (40÷90) MeV is ~ 1.6 m (9). Assuming the validity of this value also for the

(7) L. S. OSBORNE: *Phys. Rev.*, **102**, 296 (1956).

(8) D. FOURNET DAVIS: *Phys. Rev.*, **106**, 816 (1957).

(9) M. BALDO CEOLIN, M. CRESTI, N. DALLAPORTA, M. GRILLI, L. GUERRIERO, M. MERLIN, G. A. SALANDIN and G. ZAGO: *Nuovo Cimento*, **5**, 402 (1957).

energy region of $(0 \div 40)$ MeV, it is expected that on the average, one inelastic scatter happens in every 200 K^+-meson tracks in this energy region. We then assume all scatterings at 30 MeV and lower to be elastic ones in our case.

3. – Evidence of the K^+-meson nuclear scattering.

3·1. *Scattering by point nuclei.* – To know whether the K^+-meson scattering is entirely due to the Coulomb interaction, we compare the experimental angle distribution with one computed for the Coulomb scattering only. As we are dealing with K^+-mesons the kinetic energy of which is about 10% of the total energy or less, all computations have been made for the non-relativistic limit.

We first consider the scattering of the K^+-meson to be due to «point» nuclei. It is not possible to distinguish with which complex nucleus the K^+-meson collides separately, therefore the experimental cross-sections represent the resultant values. The theoretical cross-sections for the K^+-meson scattering have been computed with the aid of the Rutherford scattering formula for the kinetic energy of 30 MeV. The resultant differential cross-section $(d\sigma/d\omega)_{p,r}$ is obtained by relation (1) [10].

$$(1) \qquad (d\sigma/d\omega)_{p,r} = \sum \alpha_i (d\sigma/d\omega)_{p,i}\,, \qquad \text{with } \sum \alpha_i = 1,$$

in which $(d\sigma/d\omega)_{p,i}$ is the differential cross-section for the collision of the K^+-meson with nucleus i of the photographic emulsion, which is considered as point nucleus. For reasons given earlier [10,11], $i =$ C, Ag, Br, and O.

The curve $(d\sigma/d\omega)_{p,r} = f(\vartheta)$ for the laboratory system is drawn in Fig. 2 where also the experimental results for the energy region

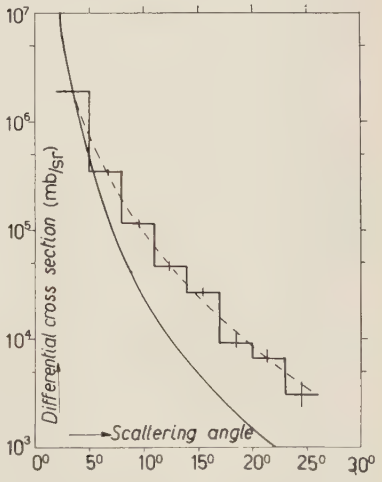

Fig. 2. – Comparison of the experimental cross-sections with those of the Rutherford scattering (solid continuous curve).

[10] T. G. Lim and P. J. van der Linden: *Nuovo Cimento*, **11**, 67 (1959).

[11] A. H. Rosenfeld, M. Backus, J. Friedman, W. F. Fry, D. Haskin, J. Lach R. Lux, M. Orans, J. Orear E. Silverstein, W. Slater, F. Solmitz, R. Swanson and H. Taft: *How to Devilup Emulsion* (Chicago, 1955), chap. II.

(0÷60) MeV are shown. A marked deviation of the experimental values indicates that the K^+-meson scattering is not entirely caused by Coulomb repulsive forces but part of the scattering should be due to nuclear interaction (2).

3·2. *Scattering by extended nuclei.* – In reality, the scattering centres are nuclei of finite dimensions. A method resulting from the first Born approximation states that if f_p is the amplitude due to a point nucleus, then the scattering amplitude due to the elastic scattering by an extended nucleus, f_e, can be found by the relation $f_e = f_p F(q)$.

The factor $F(q)$ is the so-called form factor or structure factor, which is usually given as a function of the momentum transfer $\hbar q$. Accordingly, the differential cross-section reads ([4,5])

$$(\mathrm{d}\sigma/\mathrm{d}\omega)_e = (\mathrm{d}\sigma/\mathrm{d}\omega)_p F^2(q) \,, \tag{2}$$

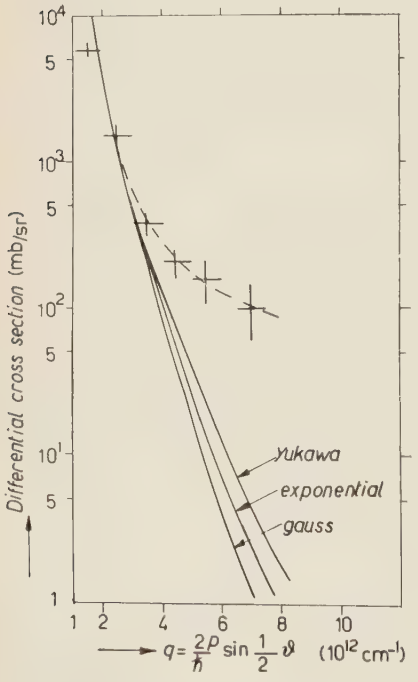

Fig. 3. – Comparison of the experimental cros ssections with those due to the extended nuclei of the Yukawa, exponential and Gaussian form.

where $(\mathrm{d}\sigma/\mathrm{d}\omega)_e$ is the differential cross-section due to the extended nucleus, and $(\mathrm{d}\sigma/\mathrm{d}\omega)_p$ is the cross-section due to the point nucleus. $\hbar q = 2\,p \sin \frac{1}{2}\vartheta$ is the momentum transfer of the particle with momentum $\boldsymbol{p}$, deflected at an angle ϑ.

In our case, the condition for the validity ([12]) of the Born approximation is satisfactorily fulfilled by the light elements, while the condition is not strictly fulfilled by the heavy elements. In spite of this, valuable qualitative information can still be obtained by applying (2). It will in the following lead to the detection of the nuclear interaction and the determination of the sign of it.

3·2.1. Detection of the nuclear scattering. – The theoretical differential cross-sections for the K^+-meson Coulomb scattering by extended nuclei are computed with the aid of relation (2). For F we take the normalized Yukawa exponential, and Gauss charge distribution functions

([12]) N. F. MOTT and H. S. W. MASSEY: *Theory of Atomic Collisions*, II ed. (Oxford, 1949), p. 125.

as given by HOFSTADTER [13]. The resultant cross-sections in the laboratory system as a function of the momentum transfer are drawn in Fig. 3.

The theoretical curves are compared with the experimental cross-sections for the K^+-mesons of kinetic energy $\lesssim$ 60 MeV. The pronounced deviation of the experimental values of the cross-sections from the theoretical values indicates the presence of the nuclear scattering.

3·2.2. Sign of the nuclear potential for the K^+-meson. – The sign of the nuclear potential can be determined by investigating whether there is a destructive or a constructive interference between the Coulomb scattering and the nuclear scattering. The nuclear scattering is assumed to depend on the potential V, which we take real (see [6]).

The point nucleus scattering amplitude f_p due to a Coulomb and a nuclear potential simultaneously, is presented by relation (3)

$$f_p = -\frac{2\mu}{\hbar^2}\left(\frac{Ze^2}{q^2} \pm \frac{R^3 V}{3}\right), \tag{3}$$

where $R = r_0 A^{\frac{1}{3}}$, R: nuclear radius, A: mass number, $r_0 = 1.2$ fermi, μ: reduced mass.

The differential cross-section for the extended nucleus is then

$$\left(\frac{\mathrm{d}\sigma}{\mathrm{d}\omega}\right)_e = \left(\frac{2\mu}{\hbar^2}\right)^2 \left(\frac{Ze^2}{q^2} \pm \frac{R^3 V}{3}\right) F^2 . \tag{4}$$

The « + » sign corresponds to a constructive interference, and the « — » sign corresponds to a destructive interference between the Coulomb and the nuclear scattering. The actual form of the nuclear potential is not known, and in the present case in which the sign of the nu-

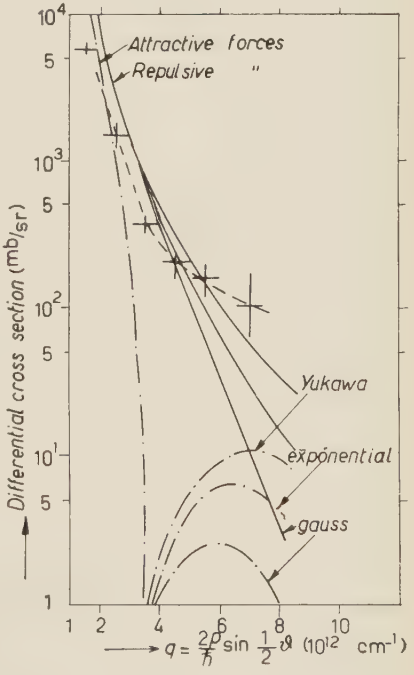

Fig. 4. – Comparison of the experimental cross-sections with the theoretical values. The constructive interference between the Coulomb and the nuclear potential ($V = +6$ MeV) is given by ——, while the destructive interference ($V = -6$ MeV) is denoted by - · - · -.

[13] R. HOFSTADTER: *Rev. Mod. Phys.*, **28**, 214 (1956).

clear potential will be determined, it is also not important. So the same form factors as the ones which are assumed to be valid for the charge distribution of the nucleus (see 3·2.1) are used for the nuclear potential.

In Fig. 4 are presented the experimental cross-sections and the theoretical curves for the laboratory system.

We see that a repulsive nuclear potential is to be favoured rather than an attractive one.

We will note that the determination of the sign of the nuclear potential has been the subject of investigation in several laboratories. The earlier reports were contradicting ([7,14,15]). However, later investigations in the energy region lower than ~ 150 MeV, all lead to the conclusion that the nuclear potential should be repulsive ([16-19]).

* * *

This work is part of the research program of the Netherlands Foundation for Fundamental Research on Matter (F.O.M.), financially supported by the Netherlands Institution for Pure Scientific Research (Z.W.O.). The authors wish to thank Dr. R. W. BIRGE, Dr. E. J. LOFGREN, and the Bevatron staff in Berkeley for the stacks exposed to the K^+-meson beam. Many thanks are due to Professor Dr. G. W. RATHENAU for the discussions and constructive criticisms, to Mr. A. G. C. TENNER for his suggestions, to Mr. D. A. SASTRADIWIRIA for the numerical computations, to Mrs. H. A. C. BARREVELD, Mr. T. J. VAN DER LINDE, and Mrs. H. E. VAN VLIET for the scanning work.

([11]) L. S. OSBORNE: *Proc. of the Sixth Annual Rochester Conference* (1956), **6**, p. 15.
([15]) L. ALVAREZ: *Proc. of the Seventh Annual Rochester Conference* (1957), **7**, p. 6.
([16]) G. COSTA and G. PATERGNANI: *Nuovo Cimento*, **5**, 448 (1957).
([17]) C. MARCHI, G. QUARENI, A. VIGNUDELLI, G. DASCOLA and S. MORA: *Nuovo Cimento*, **5**, 1790 (1957).
([18]) G. IGO, D. G. RAVENHALL, J. J. TIEMANN, W. W. CHUPP, G. GOLDHABER, S. GOLDHABER, J. E. LANNUTTI and R. M. THALER: *Phys. Rev.*, **109**, 2133 (1958).
([19]) M. A. MELKANOFF, O. R. PRICE, D. H. STORK and H. K. TICHO: *Phys. Rev.*, **113**, 1303 (1959).

RIASSUNTO (*)

Il segno del potenziale nucleare per il mesone K^+ viene determinato dalla distribuzione dello scattering elastico entro un angolo ristretto nell'emulsione nucleare G-5. La ricerca è stata eseguita nella regione di bassa energia cinetica (< 60 MeV). È stata applicata l'approssimazione di Born per analizzare i dati e si è trovato che il potenziale nucleare è repulsivo.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

Some Effects of Hole-Hole Interactions in Systems of Fermions (*).

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(ricevuto il 30 Maggio 1960)

Summary. — A generalized reaction matrix of the Brueckner theory is discussed involving the effects of hole-hole interactions in large systems of fermions. Corrections to the real part of the single particle excitation energy are estimated for the cases of nuclear matter and liquid ^{3}He. In accordance with a suggestion by Klein and Prange such corrections to the potential near the Fermi level may be large while the total energy is enhanced only a little. The effects are sentitive to nuclear density. The problem of the damping factor (*i.e.*, the imaginary energy shift) is discussed.

In a number of recent publications the effect of the motion of holes in a system of fermions has been treated ([1-4]). In particular, the Brueckner theory of nuclear matter has been modified as to take account of the hole-hole interactions ([2,3]). These interactions can be introduced as straightforward consequences of the exact treatment of the exclusion principle.

Galitskii ([1]) using methods of field theory constructs a vertex function analogous to the Brueckner reaction matrix which, however, involves the hole-

(*) This work was supported in part by the United States Air Force under contract no. AF-49 (638)-327 monitored by the Air Force Office of Scientific Research of the Air Research and Development Command.

([1]) V. M. Galitskii: *Žu. Éksp. Teor. Fiz.*, **34**, 151 (1958); A. Klein and R. Prange: *Phys. Rev.*, **112**, 994, 1008 (1958); W. Wild: *Zeits. f. Phys.*, **158**, 322 (1960).

([2]) F. Iwamoto: *Prog. Theor. Phys.*, **22**, 903 (1959) and *Prog. Theor. Phys.*, **23**, 871 (1960).

([3]) J. S. R. Chisholm and E. J. Squires: *Nuclear Phys.*, **13**, 156 (1959).

([4]) K. Gottfried and L. Pičman: *Sound Propagation in a Dilute Fermi Gas at Zero Temperature*, preprint.

hole interaction diagrams and a damping effect. The effects of holes follow from the fact that the single particle propagator (Green's function) is of the form:

$$G_0(\boldsymbol{p}, \varepsilon) = (\varepsilon - \varepsilon_p^0 + i\eta\,\Theta(p))^{-1}, \tag{1}$$

where

$$\varepsilon_p^0 = \hbar^2 p^2/2M, \quad \text{and} \quad \Theta(p) = 1 - 2n_p = \begin{cases} 1 \text{ for } |\boldsymbol{p}| > k_F \\ -1 \text{ for } |\boldsymbol{p}| \leqslant k_F ; \end{cases}$$

n_p is the occupation factor in the Fermi sea; k_F is the Fermi momentum.

IWAMOTO ([2]) gives a different formulation which does not introduce the damping effect.

Let us first examine which are the consequences of assuming the Galitskii ([1]) vertex function Γ to be a modified Brueckner reaction matrix t (*):

$$\langle \boldsymbol{k}|t|\boldsymbol{k}\rangle_{\boldsymbol{K}} = \langle \boldsymbol{k}|v|\boldsymbol{k}\rangle + \int \langle \boldsymbol{k}|v|\boldsymbol{k}''\rangle \frac{N(\boldsymbol{k}'', \boldsymbol{K})}{e_k - e_{k''} - i\eta N(\boldsymbol{k}'', \boldsymbol{K})} \cdot \langle \boldsymbol{k}''|t|\boldsymbol{k}\rangle_{\boldsymbol{K}} \mathrm{d}\boldsymbol{k}'' . \tag{2}$$

Eq. (2) follows from eqs. (14) and (16) of ref. ([1]). Here $e_k = \hbar^2 k^2/M$ and $e_{k''} = \hbar^2 k''^2/M$ are the energies of relative motion in the initial and intermediate states, respectively (in general, M should be replaced by a selfconsistent effective mass $M^*(\boldsymbol{k}, \boldsymbol{K})$); $\boldsymbol{k}$ and $\boldsymbol{K}$ are the relative and total momenta of the interacting pair, and $N(\boldsymbol{k}'', \boldsymbol{K}) = 1 - n_{(\boldsymbol{K}/2)+\boldsymbol{k}''} - n_{(\boldsymbol{K}/2)-\boldsymbol{k}''}$.

Eq. (2) differs from the Brueckner K-matrix equation by the contribution of two hole-intermediate states and by the infinitesimal imaginary shift in the energy denominator instead of the principal value.

In their most recent note MOSZKOWSKI and SESSLER ([5]) applied the separation method of MOSZKOWSKI and SCOTT ([6]) to estimate the hole-hole contribution of the real part of eq. (2). As a result of crude approximations they estimate this effect on the binding energy per nucleon to be of the order of .5 MeV, the same order of magnitude as the effect of the usual exclusion principle operator in the Brueckner theory. In the following we shall present more detailed expressions for the real part and discuss the problem of the damping factor, *i.e.*, the imaginary part of the quasiparticle energy.

One possibility of an iteration procedure for eq. (2) is to use the free two-particle t_0-matrix (compare ref. ([7])), with the standing wave boundary con-

(*) We write out only the diagonal elements which are of interest in the problem of infinite media.

([5]) S. A. MOSZKOWSKI and A. M. SESSLER: *Nuclear Phys.*, in press.

([6]) S. A. MOSZKOWSKI and B. L. SCOTT: *Ann. of. Phys.*, in press.

([7]) J. SAWICKI and J. DABROWSKI: *Nuclear Phys.*, in press.

dition (*i.e.* the principal value (P) condition; t_0 is then real). The matrix t_0 is given by:

$$\langle \boldsymbol{k}|t_0|\boldsymbol{k}\rangle = \langle \boldsymbol{k}|v|\boldsymbol{k}\rangle + P\int \langle \boldsymbol{k}|v|\boldsymbol{k}''\rangle \frac{1}{e_k - e_{k''}} \cdot \langle \boldsymbol{k}''|t_0|\boldsymbol{k}\rangle \, \mathrm{d}\boldsymbol{k}'' \,. \tag{3}$$

An equation corresponding to Galitskii's eq. (22) follows in the form:

$$\langle \boldsymbol{k}|t|\boldsymbol{k}\rangle_{\boldsymbol{K}} = \langle \boldsymbol{k}|t_0|\boldsymbol{k}\rangle + \int \langle \boldsymbol{k}|t_0|\boldsymbol{k}''\rangle \cdot \left\{ \frac{N(\boldsymbol{k}'', \boldsymbol{K})}{e_k - e_{k''} + i\eta N(\boldsymbol{k}'', \boldsymbol{K})} - P\frac{1}{e_k - e_{k''}} \right\} \langle \boldsymbol{k}''|t|\boldsymbol{k}\rangle_{\boldsymbol{K}} \, \mathrm{d}\boldsymbol{k}'' \,. \tag{4}$$

Let us denote by t_1 the result of the first iteration of eq. (4), *i.e.*, the right-hand side of eq. (4) with t replaced by t_0. We can write out the real and imaginary parts of t_1 as:

$$\mathrm{Re}\,\langle \boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}} = \langle \boldsymbol{k}|t_0|\boldsymbol{k}\rangle + P\int \langle \boldsymbol{k}|t_0|\boldsymbol{k}''\rangle \frac{N(\boldsymbol{k}'', \boldsymbol{K}) - 1}{e_k - e_{k''}} \cdot \langle \boldsymbol{k}''|t_0|\boldsymbol{k}\rangle \,, \tag{5a}$$

$$\mathrm{Im}\,\langle \boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}} = -\frac{\pi k M}{2\hbar^2} \int \mathrm{d}\Omega_{\boldsymbol{k}''} N^2\left(k\frac{\boldsymbol{k}''}{k''}, \boldsymbol{K}\right) \left\langle \boldsymbol{k}|t_0|k\frac{\boldsymbol{k}''}{k''}\right\rangle \cdot \left\langle k\frac{\boldsymbol{k}''}{k''}|t_0|\boldsymbol{k}\right\rangle . \tag{5b}$$

Let us denote by « Δ_{hh} » the respective contributions of the hole-hole interactions of these quantities ($N = -1$, $|(\boldsymbol{K}/2) \pm \boldsymbol{k}''| \leqslant k_F$):

$$\Delta_{hh}\,\mathrm{Re}\,\langle \boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}} = -\frac{M}{\hbar^2} P\int_{|(\boldsymbol{K}/2)\pm\boldsymbol{k}''|\leqslant k_F} \langle \boldsymbol{k}|t_0|\boldsymbol{k}''\rangle \frac{1}{k^2 - k''^2} \cdot \langle \boldsymbol{k}''|t_0|\boldsymbol{k}\rangle \, \mathrm{d}\boldsymbol{k}'' \,, \tag{6a}$$

$$\Delta_{hh}\,\mathrm{Im}\,\langle \boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}} = -\frac{\pi k M}{2\hbar^2} \int_{|(\boldsymbol{K}/2)\pm k(\boldsymbol{k}''/k'')|\leqslant k_F} \mathrm{d}\Omega_{\boldsymbol{k}''} \left\langle \boldsymbol{k}|t_0|k\frac{\boldsymbol{k}''}{k''}\right\rangle \cdot \left\langle k\frac{\boldsymbol{k}''}{k''}|t_0|\boldsymbol{k}\right\rangle . \tag{6b}$$

We shall now follow the method and notation of our recent paper on the optical model potential ([7]). Let the particles of the interacting pair have momenta $\boldsymbol{k}_0$ and $\boldsymbol{k}_1$, and spin variables σ_0 and σ_1, respectively. We shall calculate the contributions of the hole-hole interactions to the excitation energy of the particle « 0 ». The quantity $\langle \boldsymbol{k}|t_0|\boldsymbol{k}''\rangle\langle \boldsymbol{k}''|t_0|\boldsymbol{k}\rangle$ has the meaning of an appropriate antisymmetrized average over the isotopic spin states. In the

case of nuclear matter it can be written as:

$$(7)\qquad A(\boldsymbol{k},\boldsymbol{k}'')\equiv\frac{1}{4}\mathrm{Tr}_{(\tau_0,\sigma_1)}\left[\frac{3}{4}\langle\boldsymbol{k}|t_0|\boldsymbol{k}''\rangle_{T=1}\langle\boldsymbol{k}''|t_0|\boldsymbol{k}\rangle_{T=1}+\right.$$
$$\left.+\frac{1}{4}\langle\boldsymbol{k}|t_0|\boldsymbol{k}''\rangle_{T=0}\langle\boldsymbol{k}''|t_0|\boldsymbol{k}\rangle_{T=0}\right].$$

This function A can be, in general, expanded as $A(\boldsymbol{k},\boldsymbol{k}'')=\sum_L A_L(k,k'')\cdot P_L(\cos\vartheta)$, where ϑ is the angle between $\boldsymbol{k}$ and $\boldsymbol{k}''$. We shall only retain the (largest) isotopic term $A_0=A_0(k,k'')$. In the schematic case of spinless particles interacting in S-states only with the $\langle k|t_0|k''\rangle$ matrix, $A_0=\langle k|t_0|k''\rangle\cdot\langle k''|t_0|k\rangle$. If the separation method of ref. [6] is used then the expressions of eqs. (6) have to be replaced by ones of the same form but with t_0 replaced by t_s+v_l in the notation of ref. [6] (sum of the «short-range» t-matrix and the «long-range» two-body potential). The separation method gives a generally much better convergence of the iteration procedure at low momenta of the particles involved.

The angular integration of eq. (6) involves the condition that both the intermediate momenta lie within the domain bounded by the interesection of the two Fermi spheres, *i.e.*, where the condition $(K^2/4)+k''^2\leqslant k_F^2$ holds.

In certain intervals of the variables K and k'' the quantity $|\cos\theta|=|\boldsymbol{K}\cdot\boldsymbol{k}|/Kk''$ is bounded by $(-\alpha_0)=(k_F^2-(K^2/4)-k''^2)/Kk''<1$. We can write:

$$(8)\qquad \Delta_{hh}\,\mathrm{Re}\,\langle\boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}}=-4\pi\frac{M}{\hbar^2}P\int_0^{\sqrt{k_F^2-(K^2/4)}}\min\{1,-\alpha_0\}A_0(k,k'')\cdot\frac{k''^2\,\mathrm{d}k''}{k^2-k''^2},$$

where $\min\{1,-\alpha_0\}$ means the smaller of the values 1 and $-\alpha_0$. In fact, the condition: $(-\alpha_0)<1$ is satisfied in the interval $k_F-(K/2)<k''<\sqrt{k_F^2-(K^2/4)}$.

The corresponding contribution to the real part of the potential energy of the particle «0» is given by:

$$(9)\qquad \Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0)=4\int_{k_1\leqslant k_F}\Delta_{hh}\,\mathrm{Re}\,\langle\boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}}\,\mathrm{d}\boldsymbol{k}_1.$$

Following ref. [7] we shall now introduce the new variables of integration: k and $u=K/2$. We have $\int\mathrm{d}\boldsymbol{k}_1=(16\pi/k_0)\int k\,\mathrm{d}k\int u\,\mathrm{d}u$. The lower limit of the u-integration is $|k_0-k|$, and the upper limit is k_0+k for $0\leqslant k\leqslant\frac{1}{2}(k_F-k_0)$, and equals $\sqrt{\frac{1}{2}(k_0^2+k_F^2)-k^2}$ for $\frac{1}{2}(k_F-k_0)\leqslant k\leqslant\frac{1}{2}(k_F+k_0)$ (we consider only $k_0<k_F$). The factor 4 in eq. (9) corresponds to two different spin and charge

states of particle « 1 » in the case of nuclear matter. The right-hand side of eq. (8) is a function of k and u only, and we can denote it by $\Delta_{hh}\,\mathrm{Re}\,t_1(k,u)$. We can rewrite now eq. (9) in the form:

$$(10)\qquad \Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0) = 4\,\frac{16\pi}{k_0}\left\{\int_0^{\frac{1}{2}(k_F-k)} k\,\mathrm{d}k \int_{|k_0-k|}^{k_0+k} \Delta_{hh}\,\mathrm{Re}\,t_1(k,u)u\,\mathrm{d}u + \right.$$

$$\left. + \int_{\frac{1}{2}(k_F-k_0)}^{\frac{1}{2}(k_F+k_0)} k\,\mathrm{d}k \int_{|k_0-k|}^{\sqrt{\frac{1}{2}(k_0^2+k_F^2)-k^2}} \Delta_{hh}\,\mathrm{Re}\,t_1(k,u)u\,\mathrm{d}u\right\}.$$

On introducing now the dimensionless variables: $x = k/k_F$, $y = u/k_F$, $z = k''/k_F$, and $\nu = k_0/k_F$ we can write explicitly:

$$(11)\qquad \Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0) = -\,4\,\frac{(16\pi)^2}{4\nu}\left(\frac{\hbar^2 k_F^2}{M}\right)^{-1}\left\{\int_0^{(1-\nu)/2} x\,\mathrm{d}x \int_{|\nu-x|}^{\nu+x} y\,\mathrm{d}y + \int_{(1-\nu)/2}^{(1+\nu)/2} x\,\mathrm{d}x \int_{|\nu-x|}^{\sqrt{(1+\nu^2)/2-x^2}} y\,\mathrm{d}y\right\}\cdot$$

$$\cdot\left[\int_0^{1-y} k_F^6 A_0(k_F x, k_F z)\,\frac{z^2\,\mathrm{d}z}{x^2-z^2} + \frac{1}{2y}\int_{1-y}^{\sqrt{1-y^2}} k_F^6 A_0(k_F x, k_F z)(1-y^2-z^2)\,\frac{z\,\mathrm{d}z}{x^2-z^2}\right].$$

We can see that the above quantity is rather strongly density dependent. If the approximation of the effective range theory is used, A_0 becomes constant (S-state scattering length), and $\Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0)$ (the same holds for the imaginary part) varies with density as k_F^4. This indicates the possible importance of a future selfconsistent calculation with the Brueckner reaction matrix modified by including the effect described here. The inclusion of the hole-hole interactions might also affect the density dependence of the single particle excitation spectrum, and not only the magnitude of the potential energy.

It is well known that the off-energy-shell ($x \neq z$) components of A_0 are, in general, smaller than the diagonal ($x = z$) components (compare *e.g.* ref. [6]). In order to have an order of magnitude—and upper limit estimate of $\Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0)$ we can suppress the decreasing effect of the off-energy shell propagation, and replace A_0 by $A_0(k_F x, k_F x)$. The rather tedious y- and z-integrations of eq. (11) can be carried out explicitly (see Appendix I). For the case of nuclear matter the separation method of ref. [6] was used with the (singlet, and triplet-) S-state long range potentials computed by B. Scott [8] from the Gammel-Thaler potentials. For the « normal » nuclear density with $k_F = 1.4\,f^{-1}$

[8] B. L. Scott: to be published, private communication through Dr. S. A. Moszkowski.

we find on the Fermi surface: $\Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0 = k_F) \approx -22$ MeV (*). Even though this is an overestimate by using the on-energy-shell approximation, it indicates that the hole-hole interactions may be important, and certainly cannot be expected to be less significant than the usual effect of the Pauli principle for particle-particle interactions. The corrections of the higher orbital angular momentum potentials still increase the effect. The convergence of the series of the separation method of ref. ([6]) is believed to be rapid. Unfortunately, the efforts of the author to evaluate the next term $V^{(2)}$ have proved unsuccessful in the present case.

A similar estimate has been made for the case of liquid ^{3}He. As in the paper by ABRIKOSOV and KHALATNIKOV ([9]) the effective range approximation amplitude was used with the (singlet) S-state scattering length a (here $t_0 \approx (4\pi\hbar^2/(2\pi)^3 M)a$) The value of $k_F = .8$ A$^{\circ\,-1}$ was taken (compare ref. ([10])); it is also very close to the value of ref. ([11])). We find for $a = 1.6$ A° (see ref. ([9])): $\Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0 = k_F) \approx -2.2\,M/M^*$ °K (degrees Kelvin). This gives -1.2 °K if the value of the effective mass $M^* = 1.84\,M$ given by BRUECKNER and GAMMEL ([11]) is used. The relevant value of the normal excitation energy at the Fermi surface obtained in ref. ([11]) is $E(k_F) \approx -4.2$ °K. As the kinetic energy $T_F = \hbar^2 k_F^2/2M \approx 5.1$ °K, we have the potential energy at the Fermi surface $V(k_F) \approx -9.3$ °K. Thus our estimated correction amounts to about 13% of the latter value. However, the value of a used above is not consistent with ref. ([11]). A more realistic, «equivalent», and consistent value of a remains to be found for a more realistic estimate. $\Delta_{hh}\,\mathrm{Re}\,V^{(1)}$ is just proportional to a^2.

Let us discuss now the imaginary part of the t-matrix given by eq. (6b), As a result of this imaginary shift the quasiparticles would not be stationary states, but rather they would decay. First let us again confine ourselves to the isotropic term A_0, and evaluate the integral of eq. (6a),

$$\Delta_{hh}\,\mathrm{Im}\,\langle \boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}} = -\frac{\pi k M}{2\hbar^2}\,4\pi\,\min\{1, -\alpha_0\}A_0(k, k)\,. \tag{12}$$

The corresponding damping factor is:

$$\gamma(k_0) = \mathrm{Im}\,V^{(1)}(k_0) = 4\int_{k_1 \leqslant k_F} \Delta_{hh}\,\mathrm{Im}\,\langle \boldsymbol{k}|t_1|\boldsymbol{k}\rangle_{\boldsymbol{K}}\,\mathrm{d}\boldsymbol{k}_1\,. \tag{13}$$

(*) If $k_F = 1.2\,f^{-1}$ then $\Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0 = k_F) \approx -16$ MeV.

([9]) A. A. ABRIKOSOV and I. M. KHALATNIKOV: *Žu. Éksp. Teor. Fiz.* **33**, 1154 (1957).

([10]) V. J. EMERY and A. M. SESSLER: *Phys. Rev.*, **119**, 43 (1960).

([11]) K. A. BRUECKNER and J. L. GAMMEL: *Phys. Rev.*, **109**, 1040 (1958).

We observe that in the approximation of t_1 the particle-particle interactions do not contribute to $\operatorname{Im} V^{(1)}(k_0)$, *i.e.*, $\Delta_{pp} \operatorname{Im} V^{(1)}(k_0) = 0$ as $k_0 \leqslant k_F$, and we could omit the symbol « Δ_{hh} ». Using the method and notation as above we find:

$$(14) \quad -\gamma(k_0) = 4\,\frac{16\pi}{k_0}\left\{\int_0^{\frac{1}{2}(k_F-k_0)} k\,\mathrm{d}k\,\frac{\pi k M}{2\hbar^2}\,4\pi A_0(k,k)\int_{|k_0-k|}^{k_0+k} u\,\mathrm{d}u + \right.$$

$$\left. + \int_{\frac{1}{2}(k_F-k_0)}^{\frac{1}{2}(k_F+k)} k\,\mathrm{d}k\,\frac{\pi k M}{2\hbar^2}\,4\pi A_0(k,k)\left[\int_{|k_0-k|}^{k_F-k} u\,\mathrm{d}u + \int_{k_F-k}^{\sqrt{\frac{1}{2}(k_0^2+k_F^2)-k^2}} \frac{k_F^2-u^2-k^2}{2ku}\,u\,\mathrm{d}u\right]\right\}.$$

Finally:

$$(15) \quad -\gamma(k_0) = (2\pi)^3\,\frac{16}{\nu}\left(\frac{\hbar^2 k_F^2}{M}\right)^{-1}\left\{2\nu\int_0^{(1-\nu)/2}\mathrm{d}x\,x^3(k_F^6 A_0(k_F x, k_F x)) + \right.$$

$$\left. - \int_{(1-\nu)/2}^{(1+\nu)/2}\mathrm{d}x\,x\,(k_F^6 A_0(k_F x, k_F x))\left[\frac{1}{3}\sqrt{\frac{1+\nu^2}{2}-x^2}\left(\frac{5-\nu^2}{4}-x^2\right)-\frac{1}{3}+\frac{1-\nu^2}{2}\,x+\nu x^2-\frac{2}{3}x^3\right]\right|.$$

The integrand in eq. (15) is continuous throughout the interval $0 \leqslant x \leqslant (1+\nu)/2$, and vanishes at both ends of this interval. Using the separation method of ref. ([6]) as before and the long-range potential of ref. ([8]) we find for $k_0 = k_F(\nu = 1)$: $-\gamma \approx 7.3$ MeV if $k_F = 1.4\,f^{-1}$ and ≈ 3.9 MeV if $k_F = 1.2\,f^{-1}$.

For $k_0 < k_F$ $-\gamma(k_0)$ is still larger. Not only $-\gamma$ does not vanish at the Fermi surface, but even it turns out large. This poses an immediate difficulty as the chemical potential could not be reasonably obtained (the chemical potential corresponds to the value of k_0 for which the imaginary part of the energy eigenvalue tends to zero). This difficulty does not occur in the Green's function formulation of references ([1]) and ([4]) where the vertex function of the type of eq. (2) is not exactly a t-matrix.

A similar t-matrix could be formally introduced in the Watson theory of the optical model potential. We would then obtain a similar contribution of the hole-hole interaction to the imaginary potential from the first iteration t_1 similar to « $\operatorname{Im} V_{MFM}$ » of ref. ([7]). This contribution could be expressed as:

$$(16) \qquad \Delta_{hh} \operatorname{Im} V^{(1)}(k_0) = 4\,\frac{16\pi}{k_0}\int_{\frac{1}{2}(k_0-\sqrt{2k_F^2-k_0^2})}^{\frac{1}{2}(k_0+\sqrt{2k_F^2-k_0^2})} k\,\mathrm{d}k \int_{k_0\ k}^{\sqrt{k_F^2-k^2}} \Delta_{hh} \operatorname{Im} \langle \boldsymbol{k}\,|\,t_1\,|\,\boldsymbol{k}\rangle_{K=u} u\,\mathrm{d}u\,,$$

for $k_0 \leqslant \sqrt{2}\,k_F$, and $= 0$ for $k_0 > \sqrt{2}\,k_F$. In particular eq. (16) coincides with eqs. (14) and (15) for $k_0 = k_F(\nu = 1)$. However, the following argument can

be pointed out both against the imaginary part of eq. (16), and the damping factor γ of eqs. (13)–(15). In fact, the perturbation series expansion of eq. (2) contains terms which apparently violate the Pauli principle. If there were no imaginary shift in the energy denominator, these terms would vanish upon the summation over all the states occupied in the Fermi sea for any hermitian operator r, as it was pointed out by IWAMOTO [2]. These terms, however, do not cancel out in the presence of the imaginary shift. This suggests that the t-matrix of eq. (2) is not properly defined to be used in the Brueckner theory based on the linked cluster expansion. As for the scattering problem, where $k_0 > k_F$ the t-matrix of eq. (2) produces Pauli-principle-violating terms even for the real part of the potential in absence of the imaginary shift.

In conclusion we propose to accept the form of the reaction matrix for the ground states of systems of fermions with the principal value (P) condition:

$$\langle \boldsymbol{k}|t|\boldsymbol{k}\rangle_{\boldsymbol{K}} = \langle \boldsymbol{k}|v|\boldsymbol{k}\rangle + P\int \langle \boldsymbol{k}|v|\boldsymbol{k}''\rangle \frac{N(\boldsymbol{k}'', \boldsymbol{K})}{e_k - e_{k''}} \cdot \langle \boldsymbol{k}''|t|\boldsymbol{k}\rangle_{\boldsymbol{K}} \mathrm{d}\boldsymbol{k}'' \,. \tag{17}$$

The assumptions and approximations leading to this form have been discussed by IWAMOTO [2]. In the approximation of the so-modified Brueckner theory the damping effect of the quasiparticles does not occur.

APPENDIX

If the quantity $A_0(k_F x, k_F z)$ is replaced by its diagonal value $A_0(k_F x, k_F x)$ the y- and z-integrations in eq. (11) can be carried out explicitly. This leads to a rather lengthy expression in the general case. In the case $k_0 = k_F$ ($\nu = 1$) we can write the result:

$$\Delta_{hh}\,\mathrm{Re}\,V^{(1)}(k_0 = k_F) = -(16\pi)^2 \left(\frac{\hbar^2 k_F^2}{M}\right)^{-1} \int_0^1 \mathrm{d}x\, x k_F^6 A_0(k_F x, k_F x) f(x) \,, \tag{A.1}$$

where

$$\begin{aligned} f(x) = {} & \tfrac{1}{6}\big(1 - 9x^2 + 8x^3 - (1-x^2)^{\frac{3}{2}}\big) \ln\big|\sqrt{1-x^2} - (1-x)\big| + \\ & + \tfrac{1}{3}\big(1 - 3x^2 + 2x^3 + (1-x^2)^{\frac{3}{2}}\big) \ln\big|\sqrt{1-x^2} + (1-x)\big| + \\ & + \tfrac{1}{6}\big(1 - 3x^2 - 2x^3 + (1-x^2)^{\frac{3}{2}}\big) \ln\big|\sqrt{1-x^2} + (1+x)\big| - \\ & - x^2(1+x) \ln\big|\sqrt{1-x^2} - (1+x)\big| - \tfrac{1}{3}(1-x^2)^{\frac{3}{2}} \ln 2\sqrt{1-x^2} + \\ & + 2x^2 \ln 2x - \big(\tfrac{1}{3} - x^2 + \tfrac{2}{3}x^3\big) \ln 2\,|1-x| - \big(\tfrac{1}{3} - x^2\big) \ln 2 + \\ & + \tfrac{1}{6}(1 - 5x^2)\sqrt{1-x^2} - \tfrac{1}{6}(1 + x - 5x^2 + 3x^3) \,. \end{aligned} \tag{A.2}$$

The function $f(x) \to 0$ for $x \to 1$, and so the integrand vanishes on both ends of the integration interval, which provides a check of the calculation. The x-integration can be done numerically.

RIASSUNTO (*)

Si discute una matrice generalizzata di reazione della teoria di Brueckner che interessa gli effetti delle interazioni lacuna-lacuna in grandi sistemi di fermioni. Si valutano le correzioni alla parte reale dell'energia di eccitazione di particelle singole per i casi di materia nucleare e ^{3}He liquido. Secondo un suggerimento di KLEIN e PRANGE tali correzioni al potenziale vicino al livello di Fermi possono essere grandi mentre l'energia totale aumenta solo di poco. Gli effetti risentono della densità nucleare. Si discute il problema del fattore di smorzamento (cioè, lo spostamento energetico immaginario).

(*) *Traduzione a cura della Redazione.*

On the Integral Representation of the Double Commutator.

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(ricevuto il 3 Giugno 1960)

Summary. — A new derivation of the Dyson integral representation for the double commutator is given. It is shown that the Jacobi identity might be incorporated into this integral representation by imposing further restrictions on the weight function.

1. – Introduction.

It has been shown by Dyson [1] that there exists an integral representation of the double commutator

$$D(y, z) = \langle 0 \,|\, \big[C(x_3), [B(x_2), A(x_1)]\big] \,|\, 0 \rangle \; ; \qquad y = x_1 - x_2, \; z = x_2 - x_3 \tag{1.1}$$

fulfilling all the necessary conditions, except perhaps the Jacobi identity

$$D_{CBA}(y, z) + D_{BAC}(u, y) + D_{ACB}(z, u) = 0; \qquad y + z + u = 0 \, . \tag{1.2}$$

In this paper we present another derivation of the Dyson representation, which has the advantage of proceeding along more standard lines and gives more information about the spectral function. We shall also discuss the possibility of incorporating the Jacobi identity into the Dyson representation.

2. – Spectral representations of the single commutator.

For the convenience of the reader we shall recall in this section some results to be used in the following. The spectral representation of a causal commu-

[1] F. J. Dyson: *Phys. Rev.*, **111**, 1717 (1958).

tator has been extensively studied by LEHMANN (2), JOST and LEHMANN (3) and DYSON (4). More simple representations have been obtained recently in some special cases by FAINBERG (5).

In the following we shall be concerned mainly with matrix elements of the form

$$G_\gamma(y^2, K^2, Ky) = \langle K, \gamma | \left[B\left(-\frac{y}{2}\right), A\left(\frac{y}{2}\right) \right] | 0 \rangle \,. \tag{2.1}$$

The local commutativity condition requires that $G_\gamma = 0$ for $y^2 < 0$. To exhibit this property we shall then write

$$G_\gamma(y^2, K^2, Ky) = \int_0^\infty \mathrm{d}\nu^2 \, G_\gamma(\nu^2, K^2, Ky) \, \delta(y^2 - \nu^2) \,. \tag{2.2}$$

Taking into account the identity

$$\int_0^\infty \mathrm{d}\lambda^2 \bar{\Delta}(y^2, \lambda^2) \, \bar{\Delta}(\lambda^2, \nu^2) = (2\pi)^{-2} \, \delta(y^2 - \nu^2) \,, \tag{2.3}$$

where $\bar{\Delta}(y^2, \lambda^2)$ is the Schwinger singular function, related to the usual Δ function by the relation $\bar{\Delta}(y^2, \lambda^2) = \varepsilon(y) \, \Delta(y, \lambda^2)$, it follows immediately

$$G_\gamma(y^2, K^2, Ky) = \int_0^\infty \mathrm{d}\lambda^2 \, g_\gamma(\lambda^2, K^2, Ky) \, \Delta(y, \lambda^2) \,, \tag{2.4}$$

where

$$g_\gamma(\lambda^2, K^2, Ky) = (2\pi)^2 \int_0^\infty \mathrm{d}\nu^2 \, G_\gamma(\nu^2, K^2, Ky) \, \varepsilon(Ky) \, \bar{\Delta}(\lambda^2, \nu^2) \,. \tag{2.5}$$

The Fourier transform of (2.2) is

$$\tilde{G}_\gamma(q) = \int \mathrm{d}_4 y \, \exp[iqy] \, G_\gamma(y^2, K^2, Ky) = \int \mathrm{d}\lambda^2 \int \mathrm{d}_4 u \, \Phi(\lambda^2, u) \, \tilde{\Delta}(q - u, \lambda^2) \,, \tag{2.6}$$

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(2) H. LEHMANN: *Nuovo Cimento*, **11**, 242 (1954).
(3) R. JOST and H. LEHMANN: *Nuovo Cimento*, **5**, 1596 (1957).
(4) F. J. DYSON: *Phys. Rev.*, **110**, 1460 (1958).
(5) V. YA. FAINBERG: *JETP*, **36**, 1503 (1959).

where

$$\Phi(\lambda^2, u) = \int \mathrm{d}_4 y \, g_\gamma(\lambda^2, K^2, Ky) \exp[iuy] . \tag{2.7}$$

Introducing the Fourier transform of $g_\gamma(\lambda^2, K^2, Ky)$ with respect to the invariant Ky by

$$g_\gamma(\lambda^2, K^2, Ky) = \int_{-\infty}^{\infty} \mathrm{d}\alpha \exp[-i\alpha(Ky)] \tilde{g}_\gamma(\lambda^2, K^2, \alpha) , \tag{2.8}$$

we obtain from (2.7)

$$\Phi(\lambda^2, \mu) = 2\pi \int_{-\infty}^{\infty} \mathrm{d}\alpha \, \tilde{g}_\gamma(\lambda^2, K^2, \alpha) \, \delta(u - \alpha K) . \tag{2.9}$$

It has been proved [1] that $\Phi(\lambda^2, u)$ vanishes outside the domain defined by

$$\frac{K}{2} \pm u \in L^+ , \tag{2.10}$$

(L^+ is the forward light cone). (2.10) implies

$$-\frac{K_0}{2} \leqslant u_0 \leqslant \frac{K_0}{2} . \tag{2.11}$$

Therefore, from (2.9) it follows that

$$\tilde{g}_\gamma(\lambda^2, K^2, \alpha) \neq 0 \quad \text{only if} \quad -\tfrac{1}{2} \leqslant \alpha \leqslant +\tfrac{1}{2} . \tag{2.12}$$

3. – Derivation of the Dyson representation.

In the usual way we shall now suppose that a complete set of positive energy states exists, so that (1.1) may be decomposed as follows:

$$D(y, z) = 2 \operatorname{Re} \sum_\gamma \int_{(L^+)} \mathrm{d}_4 K \langle 0 | C(x_3) | K, \gamma \rangle \langle K, \gamma | [B(x_2), A(x_1)] | 0 \rangle = \tag{3.1}$$

$$= 2 \operatorname{Re} \sum_\gamma \int_{(L^+)} \mathrm{d}_4 K \langle 0 | C(0) | K, \gamma \rangle \langle K, \gamma | \left[B\left(-\frac{y}{2}\right), A\left(\frac{y}{2}\right) \right] | 0 \rangle \exp\left[iK\left(z + \frac{y}{2}\right) \right] .$$

Using (2.4) we may also write

$$D(y, z) = 2 \operatorname{Re} \sum_{\gamma} \int_{(L^+)} \mathrm{d}_1 K \, f_\gamma(K^2) \int_0^\infty \mathrm{d}\lambda^2 g_\gamma(\lambda^2, K^2, Ky) \, \Delta(y, \lambda^2) \exp\left[iK\left(z + \frac{y}{2}\right)\right], \tag{3.2}$$

where

$$f_\gamma(K^2) = \langle 0 \,|\, C(0) \,|\, K, \gamma \rangle \,. \tag{3.3}$$

Putting

$$\varrho(\lambda^2, K^2, Ky) = 2 \sum_\gamma f_\gamma g_\gamma, \tag{3.4}$$

(3.2) reads:

$$D(y, z) = \operatorname{Re} \int_0^\infty \mathrm{d}\lambda^2 \, \Delta(y, \lambda^2) \int_{(L^+)} \mathrm{d}_1 K \exp\left[iK\left(z + \frac{y}{2}\right)\right] \varrho(\lambda^2, K^2, Ky) \,. \tag{3.5}$$

Let us now perform the Fourier transformation of $\varrho(\lambda^2, K^2, Ky)$ with respect to Ky. Taking into account (2.12) we have

$$\varrho(\lambda^2, K^2, Ky) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \mathrm{d}\alpha \exp\left[-i\alpha(Ky)\right] \tilde{\varrho}(\lambda^2, K^2, \alpha) \,. \tag{3.6}$$

Substitution of (3.6) into (3.5) gives

$$D(y, z) = \operatorname{Re} \int_0^\infty \mathrm{d}\lambda^2 \, \Delta(y, \lambda^2) \int_0^1 \mathrm{d}\alpha \int_{(L^+)} \mathrm{d}_1 K \, \tilde{\varrho}_1(\lambda^2, K^2, \alpha) \exp\left[iK(z + \alpha y)\right] = \tag{3.7}$$

$$= \operatorname{Re} \int_0^\infty \mathrm{d}\lambda^2 \int_0^\infty \mathrm{d}\mu^2 \int_0^1 \mathrm{d}\alpha \, \tilde{\varrho}_1(\lambda^2, \mu^2, \alpha) \, \Delta(y^2, \lambda^2) \int_{(L^+)} \mathrm{d}_1 K \exp\left[iK(z + \alpha y)\right] \delta(K^2 - \mu^2) \,,$$

with $\tilde{\varrho}_1(\alpha) = \tilde{\varrho}(\frac{1}{2} - \alpha)$.

The last integral is, up to a constant factor, the function

$$\Delta^{(-)} = \tfrac{1}{2}(\Delta + i\Delta^{(1)}) \,, \tag{3.8}$$

of argument $z + \alpha y$ and mass μ.

TCP invariance requires now that $D(-y, -z) = D(y, z)$. To satisfy this requirement the term containing the function $\Delta^{(1)}$, after the substitution of

(3.8) in (3.7) must be left aside. The remaining term has the form

$$D(y, z) = \int_0^\infty d\lambda^2 \int_0^\infty d\mu^2 \int_0^1 d\alpha \, \psi(\lambda^2, \mu^2, \alpha) \, \Delta(y, \lambda^2) \, \Delta(z + \alpha y, \mu^2) \,, \tag{3.9}$$

where $\psi(\lambda^2, \mu^2, \alpha)$ is a function related in an obvious way to $\tilde{\varrho}_1(\lambda^2, \mu^2, \alpha)$. This is just the representation found by DYSON [1], who also proved its uniqueness.

4. – The Jacobi identity.

The starting point of any attempt to incorporate the Jacobi identity into an integral representation of the double commutator seems to be the imposing of some supplementary restrictions on the weight function $\psi(\lambda^2, \mu^2, \alpha)$. From this point of view, the Jacobi identity has the sole effect of restricting the class of functions given by (3.9). As a matter of fact, the functions $D(y, z)$ satisfying also the Jacobi identity form a subclass in the class of functions representable in the form (3.9).

To simplify the problem we have considered the case of three identical operators. In this case one may easily see from (2.1) and (2.4) that $\psi(\lambda^2, \mu^2, \alpha) - = \psi(\lambda^2, \mu^2, 1 - \alpha)$.

Substitution of (3.9) into the Jacobi identity gives an integral equation for ψ, which has a more simple form in Fourier space. In this way we have found the solution

$$\psi(\lambda^2, \mu^2, \alpha) = 2\varphi(\lambda^2, \mu^2)[\delta(\alpha) + \delta(1 - \alpha)] \,, \tag{4.1}$$

where $\varphi(\lambda^2, \mu^2)$ is a symmetric function of its arguments. By introducing (4.1) into (3.9) the latter becomes

$$D(y, z) = \int_0^\infty d\lambda^2 \int_0^\infty d\mu^2 \, \varphi(\lambda^2, \mu^2) \, \Delta(y, \lambda^2)[\Delta(z, \mu^2) - \Delta(u, \mu^2)] \,. \tag{4.2}$$

It is easy to see that this integral representation fulfils all the requirements for a double commutator, including the Jacobi identity. Unfortunately, we have not been able to answer the question whether (4.1) is the most general solution, although there are some arguments pointing to it.

Note added in proof.

It has been roved by Dr. K. Symanzik (private communication to Dr. R. Streater) that, unless the weight function satisfies some special conditions (which are not satisfied in perturbation theory), Dyson's representation is not the most general. On the other hand, Minguzzi and Streater (preprint CERN 8669/TH. 91) have criticized Fainberg's work (5). In view of this criticism the condition (2.12) appears to be only sufficient. We are greatly indebted to Dr. Streater for sending us some preprints concerning this problem. We would also like to mention that the solution (4.1) has been found independently of us by K. J. Le Couteur (*Nuovo Cimento*, **16**, 227 (1960)).

RIASSUNTO (*)

Si dà una nuova derivazione della rappresentazione integrale di Dyson per il doppio commutatore. Si mostra che l'identità di Jacobi potrebbe essere incorporata in questa rappresentazione integrale imponendo ulteriori restrizioni alla funzione peso.

(*) *Traduzione a cura della Redazione.*

On the μ-Meson Mass.

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(ricevuto il 6 Giugno 1960)

Summary. — A possibility of explaining the anomalously large mass of the μ-meson by taking account of the high singularity of the weak Fermi interaction is discussed. It is shown that a single spinor field $\overset{(w)}{\psi}$ interacting weakly with a two-component neutrino field can describe the electron and the μ-meson as its ground and excited states respectively by solving an eigenvalue problem which comes from the negativeness of the self-energy of $\overset{(w)}{\psi}$ due to the adopted weak Fermi interaction and by using an invariant cut-off method. The order of magnitude of the cut-off parameter is about the inverse of the weak coupling constant l_w ($l_w = 6.85 \cdot 10^{-17}$ cm). The possible importance of the high singularity of the Fermi interaction is emphasized. The problem of the μ-meson decay is also discussed.

1. – Introduction.

It is one of the central problems on the leptons how to understand the anomalously large mass of the μ-meson in spite of its similar properties to the electron in various respects [1-3]. The purpose of the present paper is to examine a possibility of understanding the large mass of the μ-meson from the standpoint of a unified spinor field theory. Previously one of the authors

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(1) R. P. Feynman: *CERN Conference Report* (Geneva, 1958), p. 216.

(2) J. Schwinger: *Ann. of Phys.*, **2**, 407 (1957); W. S. Cowland: *Nuclear Phys.*, **8**, 397 (1958); I. R. Gatland: *Nuclear Phys.*, **9**, 267 (1958); **14**, 205 (1959).

(3) G. Marx and K. L. Nagy: *Nuclear Phys.*, **12**, 125 (1959); G. Marx,: *Wiss., Zeits. (Jena)*, **8**, 371 (1958-59); J. L. Ferreira and Y. Katayama: *Progr. Theor. Phys.*, **23**, 776 (1960).

(H.K.) has argued that, in a certain framework, it is necessary to assume the existence of at least two four component spinor fields, $\overset{(s)}{\psi}$ (a strongly interacting field) and $\overset{(w)}{\psi}$ (a weakly interacting field) and a two-component spinor field χ (neutrino field) in addition to the electromagnetic field A_μ and the gravitational field $g_{\mu\nu}$ in order to describe the various elementary particles ([1]).

Itre mains, however, an open question whether these fields are sufficient to describe all the elementary particles. In the present paper, it is examined *whether a weakly interacting field* $\overset{(w)}{\psi}$ *can play the role of the fundamental lepton field*, namely, *whether* $\overset{(w)}{\psi}$ *can describe the electron and the μ-meson as its ground and excited states respectively.*

In order to simplify the problem, let us neglect the gravitational effect and any coupling of the $\overset{(w)}{\psi}$-field or χ-field with the $\overset{(s)}{\psi}$-field and consider the system which is described by the following Lagrangian density:

$$\mathscr{L} = -F_{\mu\nu}F_{\mu\nu} + \bar{\chi}\sigma_\mu\partial_\mu\chi - \overset{(w)}{\bar{\psi}}[\gamma_\mu(\partial_\mu - ieA_\mu) + \overset{(w)}{K}]\overset{(w)}{\psi} - \tag{1}$$
$$- l_w^2[\overset{(w)}{\bar{\psi}}\gamma_\mu(1+\xi\gamma_5)\overset{(w)}{\psi}](\bar{\chi}\sigma_\mu\chi)\,,$$

where $\bar{\chi} = \chi^*\sigma_4 = i\chi^*$, $\overset{(w)}{\bar{\psi}} = \overset{(w)}{\psi}{}^*\gamma_4$ and $(\sigma_1, \sigma_2, \sigma_3)$ are Pauli matrices. $\overset{(w)}{K}$ is the « bare mass » of the $\overset{(w)}{\psi}$-field. ξ is a mixing parameter of the vector and the axial vector couplings, the value of which will not be specified for the time being. l_w is a constant characterizing the weak interaction. When the value,

$$l_w = 6.85\cdot10^{-17}\ \text{cm} \tag{2}$$

is chosen in the natural unit system, l_w^2 corresponds to the usually adopted weak Fermi coupling constant.

First, in Section **2**, a homogeneous integral equation for the amplitude $\langle\Omega|\overset{(w)}{\psi}(x)|\Phi\rangle$ is derived by applying Gell-Mann and Low's procedure ([5]) to an inhomogeneous integral equation for $S_F(x-y) = -i\langle\Omega|T\overset{(w)}{\psi}(x)\overset{(w)}{\bar{\psi}}(y)|\Omega\rangle$, in order to get the eigenvalue problem of masses. In Section **3**, the divergent integrals which appear in this equation are treated by cutting off the modulus of the total four momentum of the intermediate states at λ in an invariant way similar to the Pauli and Rose method ([6]). The order of magnitude of the cut-off parameter λ is determined later.

The condition,

$$M < \overset{(w)}{K}\,,$$

([4]) H. Kita: *Prog. Theor. Phys. Suppl.* No. **9**, 5 (1959).

([5]) M. Gell-Mann and F. Low: *Phys. Rev.*, **84**, 350 (1951).

([6]) W. Pauli and M. E. Rose: *Phys. Rev.*, **49**, 462 (1936); R. Kawabe and H. Umezawa: *Prog. Theor. Phys.*, **4**, 461 (1949); S. N. Gupta: *Proc. Phys. Soc.*, **63**, 681 (1950).

which is required for getting the homogeneous integral equation as the eigenvalue problem of mass M, is guaranted by the fact that the self-energy of ${}^{(w)}\psi$-field due to the interaction,

$$l_w^2({}^{(w)}\overline{\psi}\gamma_\mu(1+\xi\gamma_5){}^{(w)}\psi)(\overline{\chi}\sigma_\mu\chi)$$

is negative. (The leading term is proportional to $-\lambda^5$. On the other hand, the electromagnetic self-energy is proportional to $\log\lambda$, and so the latter can be neglected, though positive, when compared with the former for sufficiently large values of λ.)

In the evaluation of the diverging integrals the terms lower than λ^4 are neglected. The eigenvalue equation thus obtained for the masses M involves ${}^{(w)}K$, λ and ξ as parameters. So, by inserting the observed values of the μ-meson and the electron mass for M in this equation, we get the relations by which the values of λ and ${}^{(w)}K$ can be determined. The values of λ and ${}^{(w)}K$ thus obtained for various values of ξ are shown in Table I. For values $0\leqslant\xi<1$, the order of magnitude of λ becomes $10^4 m_\mu$ (m_μ: μ-meson mass). On the contrary, for $\xi\geqslant 1$, there exists no desirable solution, because in this case we are led to the unacceptable situation: ${}^{(w)}K>\lambda$. We can choose, however, the value of ξ very close to the experimental value (~ 1).

Finally, in Section **4**, the μ-decay problem in the present attempt is discussed. After some remarks on the character of the eigenvalue problem adopted here, the way of describing the process $\mu\to e+\nu+\bar{\nu}$ and the forbiddeness of the process $\mu\to e+\gamma$ are explained.

Thus *it turns out that it is possible to describe the electron and the μ-meson as the ground and the excited state of the funadmental lepton field* ${}^{(w)}\psi$. The following objection against this might occur: the value of the cut-off parameter γ is too large ($\sim l_w^{-1}$). However, such a possibility that the cut-off may occur at about l_w^{-1} could not be excluded *a priori*, as long as we know the existence of the interactions which are characterized by the coupling constant l_w.

2. – Integral equation for the amplitude $\langle\Omega|{}^{(w)}\psi(x)|\Phi\rangle$.

We consider, first, an integral equation for the Green function of the ${}^{(w)}\psi(x)$-field defined by

$$S_F(x-y)=-2\langle\Omega|T{}^{(w)}\psi(x){}^{(w)}\overline{\psi}(y)|\Omega\rangle\,.$$

Let us restrict ourselves, however, to the simplest

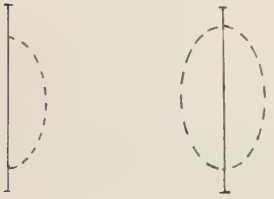

Fig. 1 – Simplest patterns ——— ${}^{(w)}\psi$-field line ---- photon line ----- neutrino line.

pattern approximation, where only the repetitions of the diagrams shown in Fig. 1 are taken into account, because it is difficult in practice to treat the exact equation without any approximation.

In this approximation, we have the following integral equation for $S_F(x-y)$:

$$
\begin{aligned}
(3)\qquad S_F(x-y) = {}& \overset{0}{S}_F(x-y) + \frac{1}{16}\iint \mathrm{d}^4z\,\mathrm{d}^4z'\,\overset{0}{S}_F(x-z')\cdot \\
& \cdot[2e^2\gamma_\mu \overset{0}{S}_F(z'-z)\gamma_\mu \overset{0}{D}_F(z'-z) + l_w^4\gamma_\varrho(1+\xi\gamma_5)\overset{0}{S}_F(z'-z)\gamma_\mu(1+\xi\gamma_5)\cdot \\
& \cdot \mathrm{Tr}\,\{\overset{(\chi)}{S}(z-z')\sigma_\varrho \overset{(\chi)}{S}(z'-z)\sigma_\mu\}]\,S_F(z-y)\,,
\end{aligned}
$$

where

$$
(4)\qquad \begin{cases}
\overset{0}{S}_F(x) = \dfrac{2}{(2\pi)^4}\displaystyle\int \mathrm{d}^4p\,\frac{\gamma_\mu p_\mu + i\overset{(w)}{K}}{p^2+\overset{(w)}{K}{}^2 - i\varepsilon}\exp[ipx]\,, \\
\overset{(\chi)}{S}_F(x) = \dfrac{2}{(2\pi)^4}\displaystyle\int \mathrm{d}^4p\,\frac{p_4\sigma_4 - (\boldsymbol{p}\cdot\boldsymbol{\sigma})}{p^2-i\varepsilon}\exp[ipx]\,, \\
\overset{0}{D}_F(x) = -\dfrac{2i}{(2\pi)^4}\displaystyle\int \mathrm{d}^4p\,\frac{1}{p^2-i\varepsilon}\exp[ipx]\,.
\end{cases}
$$

Now, by using the relation

$$
(5)\qquad S_F(x-y) = -2\sum_{\Phi}\langle\Omega|\overset{(w)}{\psi}(x)|\Phi\rangle\langle\Phi|\overset{(w)}{\bar\psi}(y)|\Omega\rangle \qquad \text{for } x_0 > y_0
$$

and applying Gell-Mann and Low's procedure [5] to the equation (3), we get the following homogeneous integral equation for the amplitude, $\varphi(x) \equiv \equiv \langle\Omega|\overset{(w)}{\psi}(x)|\Phi\rangle$, of the state Φ whose mass M ($M^2 = -P_\mu^2$, P_μ is an eigenvalue of the total energy-momentum four vector $\boldsymbol{P}_\mu$) is smaller than $\overset{(w)}{K}$.

$$
\begin{aligned}
(6)\qquad \varphi(x) = {}& \frac{1}{16}\iint \mathrm{d}^4z\,\mathrm{d}^4z'\,\overset{0}{S}_F(x-z')[2e^2\gamma_\mu \overset{0}{S}_F(z'-z)\gamma_\mu \overset{0}{D}_F(z'-z) + \\
& + l_w^4\gamma_\varrho(1+\xi\gamma_5)\overset{0}{S}_F(z'-z)\gamma_\mu(1+\xi\gamma_5)\,\mathrm{Tr}\,\{\overset{(\chi)}{S}(z-z')\sigma_\varrho \overset{(\chi)}{S}(z'-z)\sigma_\mu\}]\varphi(z)\,.
\end{aligned}
$$

By using, in the equation (6), the Fourier representation (4), the relations:

$$
(7)\qquad \varphi(x) = \langle\Omega|\overset{(w)}{\psi}(x)|\Phi\rangle = \exp[iP_\mu x_\mu]\langle\Omega|\overset{(w)}{\psi}(0)|\Phi\rangle \equiv \exp[iP_\mu x_\mu]\varphi
$$

and

$$
\begin{aligned}
(8)\quad \mathrm{Tr}\,[(p_4\sigma_4 - (\boldsymbol{p}\cdot\boldsymbol{\sigma}))\,\sigma_\varrho(p_4'\sigma_4 - (\boldsymbol{p}'\cdot\boldsymbol{\sigma}))\sigma_\mu] = {} & \\
= 2[-p_\alpha p'_\alpha\delta_{\varrho\mu} + p_\varrho p'_\mu + p'_\varrho p_\mu + \varepsilon_{\varrho\mu\lambda\sigma}p'_\lambda p_\sigma]\,, &
\end{aligned}
$$

we get the equation for φ:

$$(9)\qquad (i\gamma_\mu P_\mu + \overset{(w)}{K})\varphi = \frac{e^2}{(2\pi)^4}\int \mathrm{d}^4p\, \frac{\gamma_\varrho(\gamma_\beta p_\beta + i\overset{(w)}{K})\gamma_\varrho}{(p^2 + \overset{(w)2}{K} - i\varepsilon)\,[(P-p)^2 - i\varepsilon]}\,\varphi + \frac{2i\,l_w^4}{(2\pi)^8}\cdot$$

$$\cdot \iint \mathrm{d}^4k\,\mathrm{d}^4k'\, \frac{\gamma_\varrho(1+\xi\gamma_5)[\gamma_\beta(P_\beta + k_\beta - k'_\beta) + i\overset{(w)}{K}]\gamma'_\mu(1+\xi\gamma_5)(k_\varrho k'_\mu + k'_\varrho k_\mu - \delta_{\mu\varrho}k_\sigma k'_\sigma)}{[(P+k-k')^2 + \overset{(w)2}{K} - i\varepsilon]\,(k^2 - i\varepsilon)(k'^2 - i\varepsilon)}\,\varphi\,.$$

It is easily seen that the fourth term of the right hand side of (8) does not contribute to the integral. Furthermore, after some simple transformations, (9) reduces to:

$$(10)\qquad (i\gamma_\mu P_\mu + \overset{(w)}{K})\varphi = (J+I)\varphi\,,$$

where

$$(11)\qquad J = -\frac{2e^2}{(2\pi)^4}\int \mathrm{d}^4p'\, \frac{\gamma_\beta p'_\beta - 2i\overset{(w)}{K}}{(p'^2 + \overset{(w)2}{K} - i\varepsilon)\,[(P-p')^2 - i\varepsilon]}\,,$$

and

$$(12)\qquad I = \frac{8i\,l_w^4}{(2\pi)^8}\cdot$$

$$\cdot \iint \mathrm{d}^4k\,\mathrm{d}^4k'\, \frac{\gamma'_\mu(P_\varrho k_\mu k'_\varrho + k_\mu k_\varrho k'_\varrho + k_\varrho k_\varrho k'_\mu)(1+\xi^2 + 2\xi\gamma_5) - (i/2)\overset{(w)}{K}k_\mu k'_\mu\,(1-\xi^2)}{[(P+k-k')^2 + \overset{(w)2}{K} - i\varepsilon]\,(k^2 - i\varepsilon)(k'^2 - i\varepsilon)}\,.$$

3. – Cut-off procedure and eigenvalue problem.

Now, both integrals J and I in (10) diverge. We treat these diverging integrals by the generalized Pauli-Rose's [6] invariant cut-off method. For simplicity, let us work in the center of mass system ($\boldsymbol{P}=0$, $P_0 = M$).

i) *Integral J.* After the integration over p'_0, the integration over angles can be easily performed, because the integrand does not depend on the direction of $\boldsymbol{p}$. Next, we transform the integration variable $|\boldsymbol{p}'|$ into a new variable u' by the relation

$$(13)\qquad u' = \sqrt{\boldsymbol{p}'^2 + \overset{(w)2}{K}} + |\boldsymbol{p}'|\,,$$

u' is an invariant quantity because it is the modulus of the total momentum four-vector of the intermediate state represented (by the following diagram) in Fig. 2.

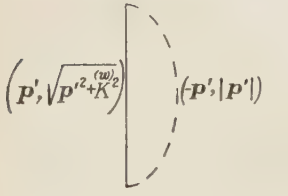

Fig. 2. – Total momentum four-vector $(\boldsymbol{P}'-\boldsymbol{p}', \sqrt{\boldsymbol{p}'^2 + \overset{(w)2}{K}} + |\boldsymbol{p}'|)$ of the intermediate state.

We cut off the contributions from the intermediate states whose u' is larger than λ ($\int_{\overset{(w)}{K}}^{\lambda} du'[\ldots]$). The integration over u' can be performed analytically. The leading terms of J are:

$$J = -\frac{2e^2}{(2\pi)^4}\left[\frac{2\pi^2 \overset{(w)}{K}}{M}\left\{\frac{M^2 - \overset{(w)2}{K}}{M}\log\left(\frac{\lambda^2 - M^2}{\overset{(w)2}{K} - M^2}\right) + \frac{2\overset{(w)2}{K}}{M}\log\left(\frac{\lambda}{\overset{(w)}{K}}\right)\right\} - \right.$$

$$\left. - \frac{\pi^2\gamma_4}{2}\left\{\frac{M^4 - \overset{(w)4}{K}}{M^3}\log\left(\frac{\lambda^2 - M^2}{\overset{(w)2}{K} - M^2}\right) + \frac{2\overset{(w)4}{K}}{M^3}\log\left(\frac{\lambda}{\overset{(w)}{K}}\right)\right\}\right]. \tag{14}$$

ii) *Integral I.* After the integrations over k_0 and k_0', we introduce the polar coordinates

$$(|\boldsymbol{k}|, \theta, \varphi) \qquad \text{for } \boldsymbol{k},$$

and

$$(|\boldsymbol{k}'|, \theta', \varphi') \qquad \text{for } \boldsymbol{k}' \ (\boldsymbol{k} \text{ is chosen as polar axis}).$$

The integrations over θ, φ and φ' can be easily performed.

Next we transform the remaining integration variables $|\boldsymbol{k}|$, $|\boldsymbol{k}'|$ and $\cos\theta'$ into new variables u, v and w by the relations

$$\begin{cases} u = \sqrt{\overset{(w)2}{K} + (\boldsymbol{k} - \boldsymbol{k}')^2} + |\boldsymbol{k}| + |\boldsymbol{k}'|, \\ v = \sqrt{\overset{(w)2}{K} + (\boldsymbol{k} - \boldsymbol{k}')^2} + |\boldsymbol{k}| - |\boldsymbol{k}'|, \\ w = \sqrt{\overset{(w)2}{K} + (\boldsymbol{k} - \boldsymbol{k}')^2} - |\boldsymbol{k}| - |\boldsymbol{k}'|. \end{cases} \tag{15}$$

All the new variables u, v and w are invariant quantities. Among them u is the modulus of the total momentum four-vector of the intermediate state represented (by the following diagram) in Fig. 3.

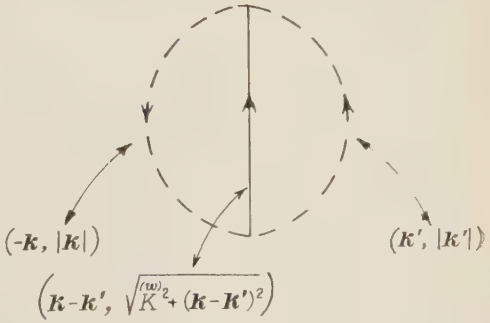

Fig. – 3. Total momentum four-vector $(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k} + \boldsymbol{k}', \sqrt{\overset{(w)2}{K} + (\boldsymbol{k} - \boldsymbol{k}')^2} + |\boldsymbol{k}| + |\boldsymbol{k}'|)$ of the intermediate state.

Thus we have

$$(16)\qquad \int_0^\infty |\boldsymbol{k}|\,\mathrm{d}|\boldsymbol{k}| \int_0^\infty |\boldsymbol{k}'|\,\mathrm{d}|\boldsymbol{k}'| \int_{-1}^{1} \frac{\mathrm{d}(\cos\theta')}{\overset{(w)}{K}{}^2 + (\boldsymbol{k}-\boldsymbol{k}')^2} \cdot [\ldots] = \frac{1}{4} \int_{\overset{(w)}{K}}^\infty \mathrm{d}u \int_{\overset{(w)}{K}{}^2/u}^{u} \mathrm{d}v \int_{v+(\overset{(w)}{K}{}^2/v)-u}^{\overset{(w)}{K}{}^2/u} \mathrm{d}w\,[\ldots]\,.$$

The integration over v and w can be performed straightforward, although lengthy. The last integration over u can be performed analytically except for the integrals of the type.

$$\int \mathrm{d}u\, \frac{\log u}{u - M}\,.$$

Here also we cut off the contributions from the intermediate states whose u is larger than λ ($\int_{\overset{(w)}{K}}^{\lambda} \mathrm{d}u[\ldots]$). The highest power with respect to λ in I is λ^5 If we neglect the terms smaller than λ^4, then we get

$$(17)\qquad I = \frac{2l_w^4}{(4\pi)^4}\left[\left\{\frac{73}{300}\lambda^5 - \frac{1}{6}M\lambda^4 \log\left(\frac{\lambda}{M}\right) + \frac{229}{720}M\lambda^4\right\}\cdot \right.$$
$$\left. \cdot\{(1+\xi^2)\gamma_4 + 2\xi\gamma_4\gamma_5\} - \frac{1}{6}(1-\xi^2)\overset{(w)}{K}\lambda^4\right]\,.$$

For sufficiently large values of λ, the electromagnetic term (14) ($\sim \log\lambda$) can be neglected in comparison to the weak interaction term (17). Thus, in this approximation, the equation (10) is reduced, in the center of mass system, to

$$(18)\qquad \left[-\gamma_4 M + \overset{(w)}{K} - 2\left(\frac{l_w}{4\pi}\right)^4 \left\{\left(\frac{73}{300}\lambda^5 - \frac{1}{6}M\lambda^4\log\left(\frac{\lambda}{M}\right) + \frac{229}{720}M\lambda^4\right)\cdot \right.\right.$$
$$\left.\left. \cdot((1+\xi^2)\gamma_4 + 2\xi\gamma_4\gamma_5) - \frac{1}{6}(1-\xi^2)\overset{(w)}{K}\lambda^4\right\}\right]\varphi = 0\,.$$

In order that the equation (18) has non-trivial solutions for φ,

$$(19)\qquad \det\left|-\gamma_4 M + \overset{(w)}{K} - 2\left(\frac{l_w}{4\pi}\right)^4 \left\{\left(\frac{73}{300}\lambda^5 - \frac{1}{6}M\lambda^4\log\left(\frac{\lambda}{M}\right) + \frac{229}{720}M\lambda^4\right)\cdot \right.\right.$$
$$\left.\left. \cdot((1+\xi^2)\gamma_4 + 2\xi\gamma_4\gamma_5) - \frac{1}{6}(1-\xi^2)\overset{(w)}{K}\lambda^4\right\}\right| = 0\,,$$

must hold.

From (19) we get:

$$(20)\qquad \left[\overset{(w)}{K}+\frac{l_w^4}{3(4\pi)^4}\overset{(w)}{K}\lambda^4(1-\xi^2)\right]^2=\left[M+\frac{2l_w^4}{(4\pi)^4}\left\{\frac{73}{300}\lambda^5-\frac{1}{6}M\lambda^4\log\left(\frac{\lambda}{M}\right)+\right.\right.$$
$$\left.\left.+\frac{229}{720}M\lambda^4\right\}(1-\xi)^2\right]\left[M+\frac{2l_w^4}{(4\pi)^4}\left\{\frac{73}{300}\lambda^5-\frac{1}{6}M\lambda^4\log\left(\frac{\lambda}{M}\right)+\frac{229}{720}M\lambda^4\right\}(1+\xi)^2\right].$$

The equation (20) is the eigenvalue equation for the mass M, if the values l_w, $\overset{(w)}{K}$, ξ and λ are specified. But we have, for the time being, no definite information about the values of $\overset{(w)}{K}$ and λ. So let us consider the inverse problem, namely, the problem of determining the values of $\overset{(w)}{K}$ and λ so that the observed masses of the μ-meson and the electron can be obtained from (20).

By inserting the μ-meson mass m_μ and the electron mass m_e for M in (20), we have the following equations:

$$(21)\qquad \left[\left(\frac{m_e}{m_\mu}\right)+2\frac{(l_wm_\mu)^4}{(4\pi)^4}\left(\frac{\lambda}{m_\mu}\right)^4\left\{\frac{73}{300}\left(\frac{\lambda}{m_\mu}\right)-\frac{1}{6}\left(\frac{m_e}{m_\mu}\right)\lg\left(\frac{\lambda}{m_\mu}\right)+\right.\right.$$
$$\left.\left.+\frac{1}{6}\left(\frac{m_e}{m_\mu}\right)\lg\left(\frac{m_e}{m_\mu}\right)+\frac{229}{720}\left(\frac{m_e}{m_\mu}\right)\right\}(1-\xi)^2\right]\left[\left(\frac{m_e}{m_\mu}\right)+2\frac{(l_wm_\mu)^4}{(4\pi)^4}\left(\frac{\lambda}{m_\mu}\right)^4\cdot\right.$$
$$\left.\cdot\left\{\frac{73}{300}\left(\frac{\lambda}{m_\mu}\right)-\frac{1}{6}\left(\frac{m_e}{m_\mu}\right)\log\left(\frac{\lambda}{m_\mu}\right)+\frac{1}{6}\left(\frac{m_e}{m_\mu}\right)\log\left(\frac{m_e}{m_\mu}\right)+\frac{229}{720}\left(\frac{m_e}{m_\mu}\right)\right\}(1+\xi)^2\right]=$$
$$=\left[1+\frac{2(l_wm_\mu)^4}{(4\pi)^4}\left(\frac{\lambda}{m_\mu}\right)^4\left\{\frac{73}{300}\left(\frac{\lambda}{m_\mu}\right)-\frac{1}{6}\log\left(\frac{\lambda}{m_\mu}\right)+\frac{229}{720}\right\}(1-\xi)^2\right]\cdot$$
$$\cdot\left[1+\frac{2(l_wm_\mu)^4}{(4\pi)^4}\left(\frac{\lambda}{m_\mu}\right)^4\left\{\frac{73}{300}\left(\frac{\lambda}{m_\mu}\right)-\frac{1}{6}\log\left(\frac{\lambda}{m_\mu}\right)+\frac{229}{720}\right\}(1+\xi)^2\right],$$

$$(22)\qquad \left(\frac{\overset{(w)}{K}}{\lambda}\right)^2=\left[1+\frac{(l_wm_\mu)^4}{3(4\pi)^4}\left(\frac{\lambda}{m_\mu}\right)^4(1-\xi^2)\right]^{-2}\cdot$$
$$\cdot\left[\left(\frac{m_\mu}{\lambda}\right)+\frac{2(l_wm_\mu)^4}{(4\pi)^4}\left(\frac{\lambda}{m_\mu}\right)^3\left\{\frac{73}{300}\left(\frac{\lambda}{m_\mu}\right)-\frac{1}{6}\log\left(\frac{\lambda}{m_\mu}\right)+\frac{229}{720}\right\}(1-\xi)^2\right]\cdot$$
$$\cdot\left[\left(\frac{m_\mu}{\lambda}\right)+\frac{2(l_wm_\mu)^4}{(4\pi)^4}\left(\frac{\lambda}{m_\mu}\right)^3\left\{\frac{73}{300}\left(\frac{\lambda}{m_\mu}\right)-\frac{1}{6}\log\left(\frac{\lambda}{m_\mu}\right)+\frac{229}{720}\right\}(1+\xi)^2\right].$$

The equations (21) and (22) for (λ/m_μ) and $(\overset{(w)}{K}/\lambda)$ have been solved numerically by using the value of l_w given by (2) and for various values of ξ. The results are listed together with the corresponding values of $(l_w\lambda/4\pi)^4$ in the Table I.

For the values $\xi \geqslant 1$, there exists no desirable solution, because in this case we are led to an unacceptable situation: $\overset{(w)}{K} > \lambda$.

From the expression (10), (17) and the whole Table I, it is seen that the self-energy of the $\overset{(w)}{\psi}$-field is negative. Accordingly, the condition $\overset{(w)}{K} > M$, which was required for getting the homogeneous equation in Section 2, is actually satisfied. Furthermore, as is seen from the Table I, we can choose the value

TABLE I.

ξ	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
$\frac{\lambda}{m_\mu}$	$2.64\cdot10^{1}$	$2.66\cdot10^{1}$	$2.75\cdot10^{1}$	$2.84\cdot10^{1}$	$3.00\cdot10^{1}$	$3.24\cdot10^{1}$	$3.54\cdot10^{1}$	$4.05\cdot10^{1}$	$4.90\cdot10^{1}$	$6.89\cdot10^{1}$
$\frac{\overset{(w)}{K}}{\lambda}$	0.155	0.158	0.173	0.185	0.209	0.246	0.290	0.367	0.492	0.745
$\left(\frac{l_w\lambda}{4\pi}\right)^4$	0.358	0.367	0.421	0.479	0.596	0.810	1.16	1.98	4.24	16.4

of the mixing parameter ξ very close to the experimental value (~ 1). This means that, if we take from the beginning the values of $\overset{(w)}{K}$ and λ which correspond to such a value of ξ, the eigenvalue equation (20) can give the observed values of the electron and the μ-meson mass as eigenvalues of M.

4. – μ-meson decay. Concluding remarks.

In the preceding section we have shown that the μ-meson can be described as an excited state of the $\overset{(w)}{\psi}$-field. Now let us examine the problem of μ-meson decay. Before going to this problem, however, it would be appropriate to review the character of the eigenvalue problem for the μ-meson mass which has been described in the above. We have treated the determination of the μ-meson mass as the real eigenvalue problem. We have neglected the terms smaller than λ^4 in Section 3. If we take into account these smaller terms also and treat the equation (6) in the momentum space, then we would have obtained a complex eigenvalue problem of M (like Gamow's method for the α-decay problem). Because the equation (6) is an homogeneous equation without any incident wave part (inhomogeneous term), and because all the Green's functions appearing in (6) satisfy the outgoing wave boundary condition. Because of the smallness of the imaginary part ($\sim$ the inverse of the life of

the μ-meson) of M, it would be necessary, in order to get a complex eigenvalue, to take into account also the very small terms, although it would make the whole calculations rather complicated.

In Section 2, however, we have neglected these small terms and treated the μ-meson mass approximately as a real eigenvalue problem.

Now, since the single $\overset{(w)}{\psi}$-field describes both the electron and the μ-meson, the field operator $\overset{(w)}{\psi}$ must be written approximately as follows:

$$\text{(23)} \qquad \begin{cases} \overset{(w)}{\psi} = \psi_e + \psi_\mu \,, \\ \overset{(w)}{\overline{\psi}} = \overline{\psi}_e + \overline{\psi}_\mu \,, \end{cases}$$

where ψ_e and ψ_μ are the annihilation operators of the electron and the μ-meson respectively and $\overline{\psi}_e$ and $\overline{\psi}_\mu$ are the corresponding creation operators. Then the third term of the original Lagrangian density (1) involves the term

$$\overline{\psi}_e[\gamma_\nu(\partial_\nu - ieA_\nu) + \overset{(w)}{K}]\psi_\mu \,.$$

At first sight, this term would seem to induce the unwanted process $\mu \to e+\gamma$. But it is easily seen that the arguments by FEINBERG *et al.* [7] and by CABIBBO *et al.* [8] hold in our present case also and the process $\mu \to e+\gamma$ can be forbidden.

The only possible real decay process of the μ-meson is induced by the term

$$\text{(24)} \qquad l_w^2(\overline{\psi}_e\gamma_\nu(1+\xi\gamma_5)\psi_\mu)(\overline{\chi}\sigma_\nu\chi) \,,$$

which is involved in the fourth term of the original Lagrangian density (1) and which is just the usually adopted weak Fermi interaction for the decay process $\mu \to e+\bar{\nu}+\nu$.

Thus we see that the weakly interacting single field $\overset{(w)}{\psi}$ can play the role of the fundamental lepton field together with the neutrino field χ. An objection might occur against the too large value ($\sim l_w^{-1}$) of the cut-off parameter λ in the Table I. Usually the cut-off parameter has been taken to be of the order of the inverse Compton wave length of the proton [9-11]. Almost all the arguments for such a value λ, however, are concerned with the strong or electro-

[7] G. FEINBERG, P. KABIR and S. WEINBERG: *Phys. Rev. Lett.*, **3**, 527 (1959).

[8] N. CABIBBO and R. GATTO: *Phys. Rev.*, **116**, 1334 (1959); N. CABIBBO, R. GATTO and C. ZEMACH: *Nuovo Cimento*, **16**, 168 (1960).

[9] G. WATAGHIN: *Zeits. f. Phys.*, **88**, 92 (1934); *Nuovo Cimento*, **5**, 689 (1957); **9**, 519 (1958).

[10] R. P. FEYNMAN and G. SPEISMAN: *Phys. Rev.*, **94**, 500 (1954).

[11] G. F. CHEW and F. E. LOW: *Phys. Rev.*, **101**, 1570, 1579 (1956).

magnetic interactions. It would not be possible at present to exclude *a priori* the above mentioned possibility of the high cut-off parameter ($\sim l_w^{-1}$), as long as we know the existence of the interactions which are characterized by the coupling constant l_w. So it may be worth-while to emphasize the possible importance of the weak Fermi interaction in the problem of self-mass ([12]).

* * *

One of the authors (H.K.) expresses his sincere thanks to Prof. G. WATAGHIN, Torino University, for the kind hospitality during his stay in Torino and for the valuable discussions. He also heartily thanks Prof. W. HEISENBERG, who made possible his visit to the Max Planck Institut für Physik und Astrophysik, for critical discussions.

([12]) In connection with this, see: H. UMEZAWA, M. KONUMA and K. NAKAGAWA: *Nuclear Phys.*, **7**, 169 (1958); H. UMEZAWA: *Prog. Theor. Phys. Suppl.* No. **7**, 69 (1959).

RIASSUNTO

Si discute un tentativo per interpretare la stranamente grande massa del mesone μ tenendo presente la forte singolarità dell'interazione debole alla Fermi. Risolvendo un problema agli autovalori che si ottiene per il fatto che la self-energia del campo $\overset{(w)}{\psi}$ derivante dall'interazione debole alla Fermi è negativa e usando un metodo invariante di cut-off, si mostra che un unico campo spinoriale $\overset{(w)}{\psi}$ interagente debolmente con un campo di neutrino a due componenti può descrivere l'elettrone e il mesone μ come suoi stati fondamentale ed eccitato rispettivamente. L'ordine di grandezza del parametro di cut-off è dell'ordine dell'inverso della costante d'accoppiamento debole l_w ($l_w = 6.85 \cdot 10^{-17}$ cm). Si discute anche il problema del decadimento del mesone μ.

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

Ion-Molecules Reactions of the Fifth Group Element Hydrides.

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(ricevuto il 7 Giugno 1960)

Summary. — Reactions of the type $M^+ + XH = MH^+ + X$, where XH stands for hydrides of the fifth group elements, or deuterium and M^+ the respective ions or A^+, have been studied. The rate constants, or their upper limits, are determined and compared with theoretical ones. Heats of formation of MH^+ ions are computed.

1. – Introduction.

Increasing attention has been given to reactions between ions and molecules in the last decade because of the role such reactions appear to play in radiation chemistry. A detailed description of previous work has been given in two symposia ([1,2]).

Reactions of this type are studied in a mass spectrometer where the pressure in the ion source is kept at such values, that secondary collisions between ion and molecules, become possible.

In this paper the experimental results of the reaction between argon, deuterium and the hydrides of the fifth group elements are presented. The results are discussed on the basis of reaction rate theory ([3]). Some thermodynamical values, of the ions formed, are also derived.

([1]) J. L. Franklin, F. H. Field and F. W. Lampe: *Advances in mass specrometry*, in *Proc. of a Conference Held in the University of London*, 24-26 Sept. 1958, edited by J. D. Waldron, p. 308.

([2]) V. L. Talrose and E. L. Frankevich: *Treatise of the First All-Union Conference on Radiation Chemistry*, p. 13, published by the Academy of Science of the U.R.S.S. (Moscow, 1958).

([3]) G. Gioumousis and D. P. Stevenson: *Journ. Chem. Phys.*, **29** 294 (1958).

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2. – Experimental Procedure [4].

The experimental work has been carried out with an Italelettronica SP21Fπ/3 sector analyzer mass spectrometer which was operated with the ion source at $+2\,000$ V the analyzer section beeing at ground potential. The geometry of the ion source is illustrated in Fig. 1. A magnetic field of 300 gauss, par rallel to the ionizing electron beam, makes the path of the electrons not diffe-significantly from its linear dimension. The trap current used in all experiments was 89 μA and the electron's accelerating voltage 40 V.

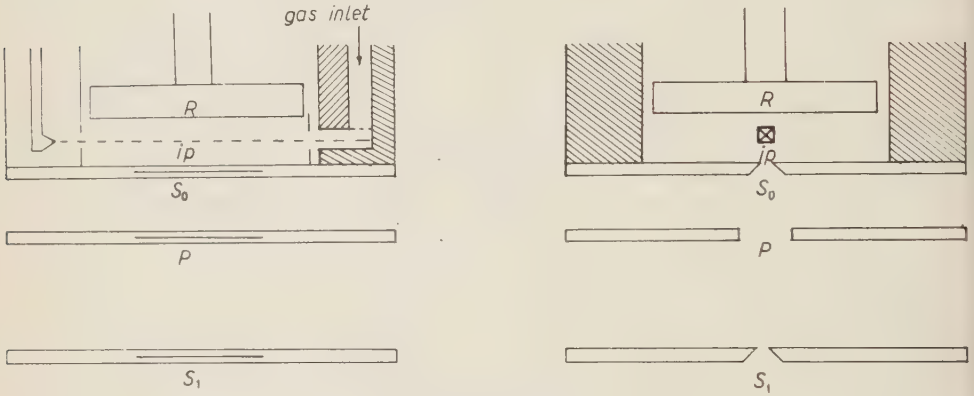

Fig. 1. – Ion source geometry: $R - S_0 = 0.325$ cm, $i_p - S_0 = l = 0.175$ cm.

The following experimental procedure was used: the studied gas was admitted from a reservoir which was at known pressure and the mass range region of interest was magnetically scanned at different values of the « repeller » voltage. Corrections for decay of pressure in the reservoir were made when necessary. The flow of gas through the « leak » which connected the reservoir to the ion source was an effusive one, except for a few experiments at higher pressure, for which corrections were made.

Since all reactions studied were of the type

$$M^+ + XH = MH^+ + X \tag{1}$$

equal efficiency of collection for the primary ion, M^+, and the secondary, MH^+, was assumed in the mass/charge region ≥ 20. The validity of the above

[4] Our experimental procedure does not differ significantly from that described by D. P. Stevenson and D. O. Schissler: *Journ. Chem. Phys.*, **29**, 282 (1958) to whom reference is made for details.

assumption is confirmed by measurements of the isotopic content in Ne, A, Kr and Xe. The assumption of constancy does not apply to light ions (mass-charge up to 6) as measurements of sensitivity for hydrogen and deuterium have shown. Therefore in this mass region semiquantitative data were obtained only.

In order to measure a phenomenological reaction cross-section for reactions (1) the pressure of XH in the ionization chamber must be known. This was done by collecting all ions produced by electron impact in He, Ne and A, on the repeller electrode. The pressure in the ion source was calculated from ion current, on the basis of known values for ionizing electron current as well as the length of the ionizing path and the absolute ionization cross-sections of the gases used ([5]). The relationship between pressures in the ion source and in the reservoir is:

$$n_{(\text{source})}\ (\text{molecules/cm}^3) = C\,(480/T)^{\frac{1}{2}}\,P_{(\text{res.})}\ (\text{micron Hg})\,,$$

$$C = (0.53 \pm 0.05)\cdot 10^{19}\ (\text{molecule/cm}^3\cdot\text{micron})\,.$$

The gases used were A, Ne, He, NH_3, PH_3, AsH_3 and D_2. Tank argon was purified on molten calcium, deuterium was prepared electrolytically from 99.7% D_2O and purified by passing it through a platinized asbestos furnace, tank ammonia ($\simeq 95\%$) was distilled bulb to bulb in vacuo. Phosphine and arsine were prepared according to ([6]) and ([7]) respectively and purified by distillation in vacuo. C.P. Helium and Neon (better than 99.8%) were used without further purification. All gases failed to show any appreciable amount of impurity ($< 0.2\%$) except for deuterium ($\simeq 3\%$ HD) and NH_3 ($\simeq 1\%$ H_2O).

3. – Experimental data.

The phenomenological cross-section for reactions (1) can be defined, on the basis of the hypothesis:

$$i_{\text{MH}^+} \ll i_{\text{M}^+}\,, \qquad i_{\text{MH}^+} \ll n_{\text{XH}}$$

as

$$Q = (i_{\text{MH}^+}/i_{\text{M}^+})(\ln_{\text{XH}})^{-1}\,, \tag{2}$$

where: i_{M^+} = intensity of M^+ ion beam,

i_{MH^+} = intensity of MH^+ ion beam,

([5]) P. T. Smith: *Phys. Rev.*, **36**, 1293 (1930).

([6]) *Inorganic Synthesis*, vol. **2** edited by V. C. Fernelius (New York, 1946), p. 141.

([7]) *Handbuch der Präparativen Anorganischen Chemie*, edited by G. Brauer (Stuttgart, 1954), p. 454.

l = path of reaction, *i.e.* distance between electron beam and exit slit,

n_{XH} = number of molecules XH, per cm³, present in the ion source.

It has been experimentally found that the phenomenological cross-section Q is a linear function of $(E_r)^{-\frac{1}{2}}$, E_r beeing the electric field which exists in the ionization chamber. It appears therefore to be convenient to define an empirical cross-section Q' of the form

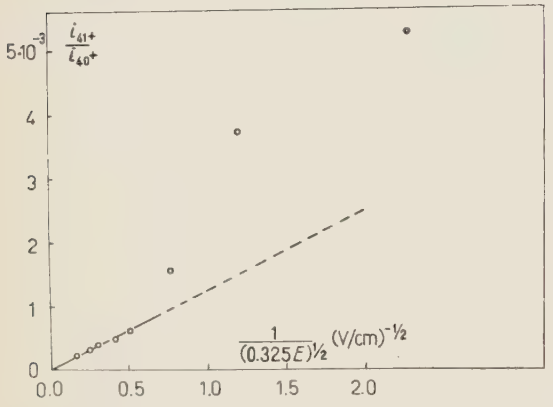

Fig. 2. – Formation of AH^- from A—H_2: P res. A = 87.5, P res. H_2 = 62.8 μm Hg.

$$Q' = Q(E_r)^{\frac{1}{2}} . \tag{3}$$

Q and consequently i_{MX^+}/i_{M^+} should be a linear function of $(V_r)^{-\frac{1}{2}}$ according to eq. (3), (V_r being the difference of potential between the repeller electrode and the exit slit), in an ideal case in which no penetrating field, space charge effects and contact potentials were present. This is not the case in our ion source, as is shown in Fig. 2, where the linear dependence shows only at values of the repeller which are greater than 9 V/cm. The rates of reaction were computed from the data belonging to the linear region. Unfortunately those data are the most affected ones, because of the small intensity of i_{MH^+}, by experimental errors in the actual readings.

3·1. *Argon-hydrogen.* – The reaction

$$A^+ + H_2 = AH^+ + H , \tag{4}$$

has been carefully studied by STEVENSON and SCHISSLER [4]. We have reproduced their experiments in order to test our apparatus. The results (two-different pressures) are given in Fig. 3.

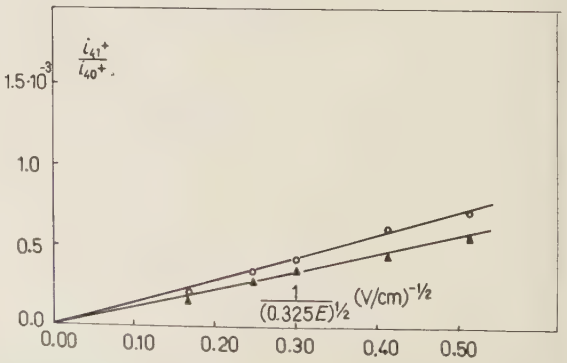

Fig. 3. – Reaction $A^+ + H_2 = AH^+ + H$: ○ P res. A = 113.0, P res H_2 = 81.6 μm Hg; ▲ P res. A = 87.5, P res. H_2 = 62.6 μm Hg.

The apparent cross section Q', obtained from the data, is $340 \cdot 10^{-16}$ cm^2/molecule (V/cm)$^{\frac{1}{2}}$.

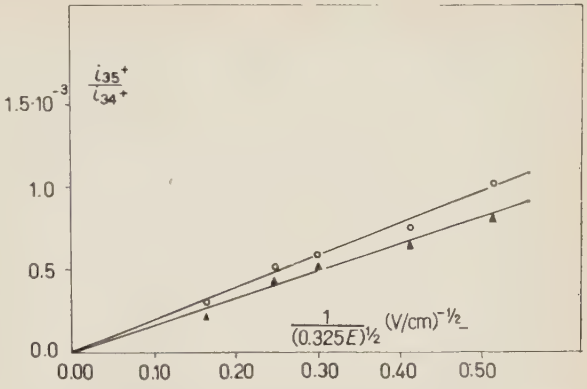

Fig. 4. – Reaction $PH_3^+ + PH_3 - PH_4^+ + PH_2$: ○ P res. $PH_3 = 192.5$ μm Hg; ▲ P res. $PH_3 = 173.5$ μm Hg.

3·2. *Formation of* XH_4^+ *ions from hydrides of the fifth group elements.* – Results for the reactions

$$(5) \quad PH_3^+ + PH_3 = PH_4^+ + PH_2 \ (?)$$

and

$$(6) \quad AsH_3^+ + AsH_3 = AsH_4^+ + AsH_2 \ (?)$$

are given in Fig. 4 and 5 and the respective Q' obtained are: $187 \cdot 10^{-16}$ and $225 \cdot 10^{-16}$.

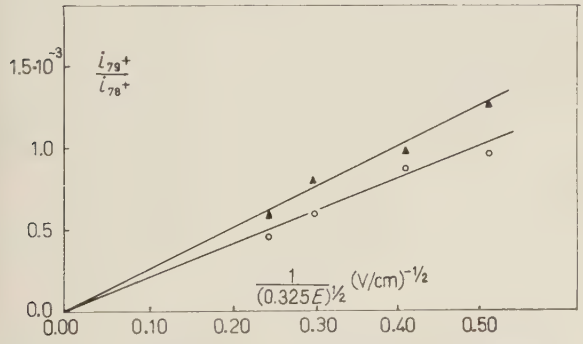

Fig. 5. – Reaction $AsH_3^+ As + H_3 = AsH_4^+ + AsH_2$: ○ P res. $AsH_3 = 231.0$ μm Hg; ▲ P res. $AsH_3 = 164.0$ μm Hg.

An attempt to measure the rate of formation of ammonium ions from ammonia has failed because of the impossibility to avoid interference of water both as background and as impurity in the gas. The rate of formation of ammonia, in the same condition was already measured by means of an indirect method ([8]).

3·3. *Reactions between argon and hydrides.* – The reaction

$$(7) \quad A^+ + XH_3 = AH^+ + XH_2 \ (?)$$

gave negative results with all three hydrides. We can assume therefore that the reactions occur, if at all, within the limits of the sensitivity of our experimental technique. The upper limit for apparent cross-section will thus be of $2 \cdot 10^{-16}$.

([8]) L. M. Dorfman and P. C. Noble: *Journ. Phys. Chem.*, **63**, 980 (1959).

3·4. *Reactions between deuterium and hydrides.* – Two types of reactions are possible in this system:

$$XH_3^+ + D_2 = XH_2D^+ + D \tag{8}$$

and

$$D_2^+ + XH_3 = D_2H^+ + XH_2 \ (?) \tag{9}$$

Reactions of type (8) were not detected and the same consideration as those mentioned for reactions (7) apply here.

For reactions of type (9) a very weak signal at mass/charge = 5 has been detected. The cross-section Q seems to follow the normal dependence, of the repeller voltage, for ion molecule reactions. The magnitude of Q', for all reactions is $5 \cdot 10^{-16}$, we wish to emphasize however that these data are obtained at the limit of our experimental errors.

4. – Thermodynamical considerations.

If an ion-molecule reaction is experimentally observed, its heat has to be $\leqslant 0$. Therefore from reactions (4), (4*a*), (5), (6) and (6*b*) an upper limit for

TABLE I (*).

Reaction	Heat of reaction ΔH kcal/mole	Heat of formation of the ion kcal/mole
(6*b*) $NH_3^+ + NH_3 = NH_4^+ + NH_2$	$\leqslant 0$	$\Delta H_f(NH_4^+) \leqslant 172$
(8) $NH_3^+ + H_2 = NH_4^+ + H$	$\leqslant 1$	—
(4*a*) $H_2^+ + H_2 = H_3^+ + H$	$\leqslant 0$	—
(9*a*) $H_2^+ + NH_3 = H_3^+ + NH_2$	$\leqslant -1$	—
(4) $A^+ + H_2 = AH^+ + H$	$\leqslant 0$	—
(7*a*) $A^+ + NH_3 = AH^+ + NH_2$	$\leqslant -1$	—
(5) $PH_3^+ + PH_3 = PH_4^+ + PH_2$	$\leqslant 0$	$\Delta H_f(PH_4^+) \leqslant 208$
(8*b*) $PH_3^+ + H_2 = PH_4^+ + H$	$\leqslant 26$	—
(9*b*) $H_2^+ + PH_3 = H_3^+ + PH_2$	$\leqslant -26$	—
(7*b*) $A^+ + PH_3 = AH^+ + PH_2$	$\leqslant -26$	—
(6) $AsH_3^+ + AsH_3 = AsH_4^+ + AsH_2$	$\leqslant 0$	$\Delta H_f(AsH_4^+) \leqslant 264$
(8*c*) $AsH_3^+ + H_2 = AsH_4^+ + H$	$\leqslant 44$	—
(9*c*) $H_2^+ + AsH_3 = H_3^+ + AsH_2$	$\leqslant -44$	—
(7*c*) $A^+ + AsH_3 = AH^+ + AsH_2$	$\leqslant -44$	—

(*) Heats of formation of ion have been taken from F. H. FIELD and J. L. FRANKLIN *Electron Impact Phenomena* (New York, 1957). Heats of formation of molecules from F. R. BICHOWSKY and F. D. ROSSINI: *The Thermochemistry of the Chemical Substances* (New York, 1936). Bond energies NH_2—H, PH_2—H, AsH_2—H are from T. L. COTTRELL: *The Strenghts of Chemical Bonds* (London, 1958), assuming for PH_2—H, AsH_2—H average values of As—H and P—H bonds in AsH_3 and PH_3 respectively. The ionization potential for AsH_3 has been assumed to be =10 V. Errors introduced by the last two assumptions should not excede ±10 kcal/mole in final heats of reactions.

the heats of formation of the ionium ions AH^+, XH_4^+ and H_3^+ can be obtained and from them the ΔH for other reactions calculated. The results are summarized in Table I. The following conclusions can be drown from those data:

a) Reactions (7*b*) and (7*c*) are energetically possible and therefore only kinetic considerations can explain the absence of experimental evidence.

b) Reactions (8*b*) and (8*c*) are endothermic unless (5) and (6) are exothermic (26 and 44 kcal/mole respectively at least). This could explain their absence in our experimental conditions.

c) Nothing conclusive can be said about reaction (8), the upper limit of the heat of reaction beeing very close to zero.

5. – Determination of the rate constants.

In radiation chemistry it is important to calculate, the rate constant for thermal conditions, where both ions and molecules have maxwellian velocity distributions, from experimental cross-sections of ion molecule reactions obtained in an electric field.

In such a field the ions, which are formed at the ionization plane, are subjected to an acceleration towards the exit slit and their velocity is not constant. This effect can be taken care of in two different ways:

a) The microscopic reaction cross-section is not relative velocity dependent. In this case it is sufficient to compute the average velocity with which an ion covers the distance l (hence the average time spent by the ion in the source) and to calculate then the rate constant in thermal conditions:

$$i_{MH^+} = d(MH^+)/dt = k c_{M^+} c_{XH} = k\, i_{M^+}(l/v) c_{XH} = k\, i_{M^+}(eEl/600 m_{M^+})^{\frac{1}{2}} c_{XH},$$

e = charge of the electron in e.s.u.

c_{M^+}, c_{XH} = concentration of M^+ and XH in the ion source in molecule/cm^3,

m_{M^+} = mass of the ion M^+ in atomic unit

$$k = Q'(eEl/600\, m_{M^+})^{\frac{1}{2}}. \tag{10}$$

b) The microscopic reaction cross-section depends on the relative velocity of the ion and the molecule. The interaction force is of the charge-induced dipole type and the rate of the reaction is calculated classically ([8,9]).

([8]) S. Glasstone, K. J. Laidler and H. Eyring: *The Theory of Rate Processes* (New York, 1941), p. 220.

The rate constant has the form of:

$$k = 2(e^2\alpha/\mu)^{\frac{1}{2}}, \tag{11}$$

α = polarizability of the molecule,

μ = reduced mass,

and its relation to Q' is the same as in *a*).

Theoretical rate constants, computed from (11), are compared in Table II with experimental ones obtained from the Q' and (10). It can be seen that the agreement between theoretical and experimental k's is fairly good for reactions (4), (5) and (6), expecially in view of the large experimental error, which can be estimated to be about 10%. Most of the error is due to uncertainties in values of the electric field in the ion source and therefore affects only the absolute values of the rate constants but not their ratios.

TABLE II (*).

Reactions	$\alpha \cdot 10^{24}$	$Q' \cdot 10^{16}$	Exp. our	$k \cdot 10^9$ Exp. others	Theor.
(4) $A^+ + H_2 = AH^+ + H$	0.79	340	1.56	1.68 [4]	1.50
(5) $PH_3^+ + PH_3 = PH_4^+ + PH_2$	4.84	187	0.93	—	1.24
(6) $AsH_3^+ + AsH_3 = AsH_4^+ + AsH_2$	5.75	225	0.72	—	0.90
(6*b*) $NH_3^+ + NH_3 = NH_4^+ + NH_2$	2.26	—	—	0.5 [8]	1.21
(7*a*) $A^+ + NH_3 = AH^+ + NH_2$	2.26	< 2	< 0.02	—	1.02
(7*b*) $A^+ + PH_3 = AH^+ + PH_2$	4.84	< 2	< 0.02	—	1,20
(7*c*) $A^+ + AsH_3 = AH^+ + AsH_2$	5.75	< 2	< 0.02	—	1.09
(8*a*) $NH_3^+ + D_2 = NH_3D^+ + D$	0.775	< 0.3	< 0.01	—	1.145
(8*b*) $PH_3^+ + D_2 = PH_3D^+ + D$	0.775	< 0.3	< 0.01	—	1.09
(8*c*) $AsH_3^+ + D_2 = AsH_3D^+ + D$	0.775	< 0.3	< 0.01	—	1.055
(9*a*) $D_2^+ + NH_3 = D_2H^+ + NH_2$	2.26	~ 5	~ 0.03	—	1.99
(9*b*) $D_2^+ + PH_3 = D_2H^+ + PH_2$	4.84	~ 5	~ 0.03	—	2.72
(9*c*) $D_2^+ + AsH_3 = D_2H^+ + AsH_2$	5.75	~ 5	~ 0.03	—	2.87

Values for α have been taken from Landolt-Börnstein: *Zahlenwerte und Funktionen* (Berlin, 1951), vol. I, part 3, for all gases except PH_3 and AsH_3 for which data from *Table of Dielectric Constants and Electric Dipole Moments of Substances in the Gaseous State*, National Bureau of Standards Circular 537 (June 25, 1953), were used.

As has already been pointed out by GIOUMOUSIS and STEVENSON [3], the experimental k's, for molecules with a permanent electric dipole, should be higher than theoretical ones which are calculated on the basis of a simple

(*) α's are expressed in cm^3, Q's in $(cm^2/mol)(V/cm)^{\frac{1}{2}}$, and k's in $(cm^3/mol\ s)$.

ion-induced dipole attractive force. It is difficult to calculate quantitatively the effect of the permanent dipole even though a net attractive effect is expected. Contrarily to this our experimental data (reactions (5) and (6)) as well as some others which are due to other workers ([4]) seem to indicate an effect in the opposite direction. It seems therefore necessary to investigate other reactions and use molecules having a permanent dipole moment to check the effect.

Reactions (7) and (8) cannot be taken as a proof of a disagreement with theory. They could either be endothermic (see Section **4**) or a more efficient process might take place in competition with reactions studied (charge exchanges have already been invoked to explain the absence of formation of AH^+ in the system $A—CH_4$ ([10])).

The presence of an activation energy would explain why some energetically possible reactions have not in fact been observed (an activation energy of $(4 \div 5)$ kcal/mole would make a reaction undetectable with present experimental techniques). Up to date there is no evidence for ion-molecule reactions with activation energies. Should such evidence become available the theoretical model would have to be drastically revised.

In conclusion one can say that investigating the effect of permanent dipole moments and that of the eventual presence of activation energies appears to be necessary in order to have a more valid basis for a comparison with theory. A better knowledge of charge exchange processes is also necessary for the understanding of the mechanism of ion-molecule reactions. On the other hand we wish to point out that experimental data are already of great help for the interpretation of the role that ions can have in radiation chemistry.

* * *

We wish to acknowledge the help of Dr. A. Mele and Mr. E. Luzzatti of this laboratory in the experimental work.

([10]) G. G. Meisels, W. H. Hamill and R. R. Williams jr.: *Journ. Chem. Phys.* **25**, 790 (1956).

RIASSUNTO

Sono state studiate reazioni del tipo $M^+ + XH = MH^+ + X$, essendo XH idruri degli elementi del quinto gruppo o deuterio e M^+ i rispettivi ioni o A^+. Le costanti di velocità, o i loro limiti superiori, sono state determinate e paragonate con le teoriche. Sono stati inoltre calcolati i calori di formazione degli ioni MH^+.

Comparison of the Mean Gap Length in Ilford G-5 Emulsion with Theory.

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(ricevuto il 7 Giugno 1960)

Summary. — This work describes ionization measurements on the tracks of particles with velocities corresponding to β between 0.25 and 0.8. The results have been obtained by the determination of the mean gap length. A theoretical curve is calculated with the help of the well-known relation between the relative grain density and the velocity as checked by Jongejans and the relation between the mean gap length and the grain density as given by Barkas. The results are in agreement with this theory.

1. – Introduction.

In order to obtain a curve giving a relation between mean gap length and velocity, use can be made of a relation between grain density and velocity calibrated at high energies. (The gap length is defined as the distance between the edges of two neighbouring developed silver bromide grains.) Assuming proportionality between grain density and specific restricted energy loss, Jongejans [1] made use of a curve calculated by Sternheimer [2] based on the theory of Bethe. (The specific restricted energy loss is the sum of all energy losses per unit length smaller than T.) Two parameters were invoked to adapt theory to experiment: the mean ionization potential I and the cut-off energy T. The value of I has been determined by Barkas [3] by means of range measurements. I has been calculated to be $=500$ eV [1] for silver bromide from

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[1] B. Jongejans: *Nuovo Cimento*, **16**, 625 (1960).

[2] R. M. Sternheimer: *a*) *Phys. Rev.*, **88**, 851 (1952); *b*) **89**, 1148 (1953); *c*) **91**, 256 (1953); *d*) **93**, 351 (1953); *e*) **93**, 1434 (1953); *f*) **103**, 511 (1956).

[3] W. H. Barkas: *Nuovo Cimento*, **8**, 201 (1958).

the value for an average emulsion nucleus. It follows from blob measurements at high energy that the most probable value of T is 100 keV [1] (*). The final result is not very sensitive to the chosen value of T/I^2.

In this work we extrapolate the relation between grain density and velocity to lower energy: $0.25 < \beta < 0.8$. There are several arguments against this extrapolation:

1) Breakdown of proportionality between grain density and specific restricted energy loss.

2) Different value of the parameter T/I^2 at low energy.

It seems to follow from our measurements (Section 3) that these effects are small. They will therefore not be taken into consideration and no shell corrections [4,3] will be applied to the formula. The density effect [5,3] need not be taken into account as this effect only occurs in the extremely relativistic region. The grain density formula of Sternheimer-Bethe will be used:

$$g \sim \frac{1}{\beta^2}\left(B + \ln \frac{\beta^2}{1-\beta^2} - \beta^2\right), \tag{1}$$

in which: β = velocity in units of the light velocity,

$B = \ln 2mc^2 \cdot T/I^2 = 12.4$ with m = the electron mass,

$T = 100$ keV, as obtained by JONGEJANS for the emulsion stack used,

$I = 500$ eV, as obtained by BARKAS.

At velocities below $\beta = 0.30$, T should be replaced by t, *i.e.* the maximum energy transfer possible to an electron:

$$t = (\gamma^2 - 1)\mu c^2/(\mu/2m + m/2\mu + \gamma) \approx 2(\gamma^2 - 1)mc^2 \tag{2}$$

(μ is the mass of the incoming particle, m that of an electron, $(\gamma = (1-\beta^2)^{-\frac{1}{2}})$.

At low energy it is not advisable to measure the ionization by counting blobs: in the neighbourhood of $\beta = 0.3$ the blob density is a very slowly varying function of the velocity. The best method is the determination of

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(*) Dr. BARKAS informed me that new measurements on heavy elements give a 20 percent lower value of I. If this new value is used we get a best fit for $T = 20$ keV.

(4) M. C. WALSKE: *Phys. Rev.*, **101**, 940 (1956).

(5) E. FERMI: *Phys. Rev.*, **57**, 485 (1940).

the mean length of the gaps between the developed grains. The problem is to find the relationship between grain density and mean gap length. The relation given by different models is:

$$g = \frac{1}{W_L - L + A}, \tag{3}$$

in which: g is equal to the grain density per unit length, W_L is equal to the mean length of the gaps larger than L in the developed emulsion, and A is a constant varying between 0 and 0.4 μm dependent on the models and the given emulsion.

In the following we shall mention three models for grain formation in nuclear emulsion. The different curves resulting from these models will be tested by experiment. This has been performed in the AF stack exposed to 5.7 GeV negative pions. A description of the properties of this stack can be found in references (1) and (6).

2. – Models giving a relation between mean gap length and grain density.

The following models will be discussed:

1) The model of Herz and Davis (7).

2) The model of O'Ceallaigh (8).

3) The model of Barkas (9).

The first two models are based on two simplifying assumptions. The first and most important is that the centres of the silver bromide crystals lie on the track of the particle. The second is that the diameters of the crystals are all equal to α. If G is the length of a gap between two neighbouring developable crystals (irradiated crystals that give grains after development), $\bar{G}$ the mean gap length, G_{tot} the total gap length per unit length in the undeveloped emulsion and g the density of the developable crystals, relations (4) and (5) can be derived:

$$G_{tot}/\bar{G} = g \tag{4}$$

and

$$G_{tot} = 1 - g\alpha \tag{5}$$

(6) D. J. Holthuizen and B. Jongejans: *Suppl. Nuovo Cimento*, **14**, 429 (1959).
(7) A. J. Herz and G. Davis: *Austr. Journ. Phys.*, **8**, 129 (1955).
(8) C. O'Ceallaigh: *Rep. Cosmic Ray Congress*, Bagnères-de-Bigorre, **73** (1953).
(9) W. H. Barkas: *U.C.R.L.*, 8687, 14 (1959).

and from this

$$g = \frac{1}{\bar{G} + \alpha} . \tag{6}$$

Formula (6) gives a relationship between the grain density and the mean gap length in the undeveloped emulsion. $\bar{G}$ is unknown. If W_L is the mean length of the gaps larger than L in the developed emulsion it follows that:

$$W_{L'} - L' = G_L - L \qquad (L > \Delta\alpha) \tag{7}$$

if $L' + \Delta\alpha = L$. G_L is the mean length of the gaps larger than L between developable crystals in undeveloped emulsion. A sufficient condition for this equation is that the growth $\Delta\alpha$ of all crystals during development is central and equal. If L is sufficiently large, no influence of the resolving power of the microscope is to be expected. The first two models give a relation between the mean gap length $\bar{G}$ and the mean length of the gaps larger than L in the undeveloped irradiated emulsion.

The model of Herz and Davis ([7]) assumes that the distance between the edges of two neighbouring silver bromide crystals is a constant β (not to be confused with the velocity β). If p is the probability of development of a crystal, the mean length of the gaps with a length larger than $L = (n+1)\beta + n\alpha$ is:

$$G_L = \frac{\sum_{j=n}^{\infty} \{(j+1)\beta + j\alpha\} p^2 (1-p)^j}{\sum_{j=n}^{\infty} p^2 (1-p)^j} = L - \beta + \bar{G} . \tag{8}$$

From (6), (7) and (8) can be derived:

$$g = \frac{1}{W_L - L + \alpha + \beta} . \tag{9}$$

In G-5 emulsion $\alpha = 0.27$ μm and the volume of the silver bromide is nearly equal to the volume of the gelatine, $\alpha + \beta$ is about 0.4 μm.

O'Ceallaigh ([8]) assumed an exponential distribution for the distances between the centres of two neighbouring crystals. In this model is:

$$G_L = \frac{\int_L^{\infty} x \exp\left[-(x-\alpha)/\bar{G}\right] \mathrm{d}x}{\int_L^{\infty} \exp\left[-(x-\alpha)/\bar{G}\right] \mathrm{d}x} = L + \bar{G} , \tag{10}$$

and from this:

$$g = \frac{1}{W_L - L + \alpha} \,. \tag{11}$$

The most promising idea comes from BARKAS [9] (*). He assumes a distribution of the diameters D of the crystals given by a function $F(D)$. The development probability is a function $1 - G(D, \delta)$ of the diameter D and of the traversal δ through the crystal. Of course it is also a function of the energy loss but this is not essential here. If the number of grains per unit volume is equal to N, the grain density is given by:

$$g = \frac{\pi N}{2} \int_0^\infty F(D)\,\mathrm{d}D \int_0^D \delta(1 - G)\,\mathrm{d}\delta \,. \tag{12}$$

The probability $q(\mu)$ that a particle, after rendering developable a crystal with diameter D, does not cause developability of a crystal with diameter D' over a subsequent distance exceeding μ is:

$$\frac{\mathrm{d} \ln q(\mu)}{\mathrm{d}\mu} = -\frac{\pi^2 N^2}{4g}\, \delta\delta'(1 - G)(1 - G')\, F(D)\, F(D')\, \mathrm{d}\delta\, \mathrm{d}\delta'\, \mathrm{d}D\, \mathrm{d}D' \,, \tag{13}$$

in which μ is the distance between the centres of the two crystals, projected on the track of the particle. After development a gap of length L is left between the two crystals with L given by:

$$L = \mu - (D + D')/2 - (\varepsilon + \varepsilon')/2 \,, \tag{14}$$

if ε is the growth of a crystal caused by the development of the emulsion. By integrating of (13) over μ formula (15) can be derived:

$$H(L) = g \exp\left[-\frac{\pi^2 N^2}{4g}\right] \int_0^\infty \mathrm{d}D \int_0^\infty \mathrm{d}D' \int_0^D \mathrm{d}\delta \int_0^{D'} \mathrm{d}\delta' \cdot \tag{15}$$

$$\cdot \{L + (D + D')/2 + (\varepsilon + \varepsilon')/2\}\, \delta\delta'(1 - G)(1 - G')\, F(D)\, F(D') \,,$$

or

$$H(L) = g \exp\left[-g D_{\mathrm{dev}}\right] \exp\left[-gL\right] , \tag{16}$$

in which D_{dev} is the mean diameter of the developed grain. $H(L)$ is an expo-

(*) I am indebted to Dr. BARKAS for correcting some mistakes I made in reproducing his theory.

nential function of L and $\mathrm{d}\ln H/\mathrm{d}L$ is equal to $-g$. Now equation (3) can be written as:

$$g = \frac{1}{W_L - L}. \tag{17}$$

3. – Measurements.

In the AF stack a large number of stopping pions and protons due to reactions of the beam pions are available. Several positive pions and one proton are selected. The pions are recognized by the pion muon decay, the proton is identified by the constant sagitta multiple scattering method. The velocity at the place of measurements follows from the relationship between energy and range as given by WILLIS *et al.* [10]. For the measurements of the length of the gaps a Leitz Ortholux monocular microscope was used with a 15× eyepiece (with micrometer) and a 100× objective. For each β about 570 gaps were measured. The measurements were made by two scanners. Some measurements were repeated.

In the calculation of $W_L - L$ four values were chosen for L. For statistical reasons a small value is preferable. If L is too small, systematical errors may be introduced. In Tables I and II the calculated values of $W_L - L$ are given as a function of L for scanner H and scanner B respectively. A value

TABLE – I. $(W_L - L)$ *in* μm *as a function of* L *and* β *as measured by scanner* H. *The percentual statistical error* Δ $(=100/\sqrt{N})$ *is given.*

β	$L=0$ (μm)	Δ	$L=0.29$ (μm)	Δ	$L=0.58$ (μm)	Δ	$L=0.77$ (μm)	Δ
0.82	3.8	4.3	3.8	4.5	3.9	4.7	3.9	4.8
0.585	2.03	4.3	2.02	4.6	2.06	5.0	2.00	5.2
0.52	1.60	4.3	1.49	4.5	1.56	5.0	1.47	5.3
0.455	1.25	4.2	1.22	4.7	1.22	5.3	1.27	5.8
0.38	0.87	4.1	0.83	4.7	0.94	5.9	0.89	6.4
0.37	0.91	4.2	0.79	4.7	0.79	5.7	0.84	6.5
0.36	0.79	4.3	0.78	5.1	0.86	6.4	0.87	7.1
0.33	0.72	4.2	0.70	5.1	0.75	6.4	0.78	7.4
0.32	0.67	5.2	0.64	6.2	0.74	8.3	0.71	9.3
0.32	0.57	4.9	0.57	6.3	0.62	8.4	0.61	9.7
0.27	0.51	5.2	0.45	6.5	0.46	8.9	0.42	11
0.27	0.49	5.2	0.46	6.9	0.47	9.4	0.43	11

[10] B. H. WILLIS, C. V. STABLEFORD and J. H. ATKINSON: *U.C.R.L.*, 2426, (rev) (1957).

of $L = 0.58$ μm seems the most promising, because for every value of β the value of $W_L - L$ seems to change systematically with increasing L up to $L = 0.58$ μm. This can especially be seen in Table II. The statistical error is taken as the square root of the number of the measured gaps, see PEIERLS [11].

TABLE II. – $(W_L - L)$ *in* μm *as a function of* L *and* β *as measured by scanner B. The percentual statistical error* Δ $(=100/\sqrt{N})$ *is given.*

β	$L=0$ (μm)	Δ	$L=0.29$ (μm)	Δ	$L=0.58$ (μm)	Δ	$L=0.77$ (μm)	Δ
0.73	3.3	4.3	3.1	4.3	3.0	4.5	3.1	4.7
0.52	1.78	4.2	1.68	4.3	1.62	4.7	1.59	4.9
0.455	1.40	4.2	1.22	4.4	1.21	5.0	1.22	5.4

The grain density, and therefore also the mean gap length, is not only a function of the velocity but also of the degree of development. To eliminate the influence of the fluctuations in the development, in each region of the measurements the ratio of the grain density in question to the grain density of the beam pions in the neighbourhood was taken. To reduce the measuring time, the counting of blobs was carried out on the beam pion tracks. At this high energy the relation between blob density and grain density can be given by the experimental formula:

$$B = g \exp[-Kg] . \tag{18}$$

TABLE III. – $(W_L - L)^{-1}$ in μm^{-1} *as a function of* L, *as measured on six beam pions. The statistical error of each measurement is given.*

$L=0$	$L=0.58$ μm	$L=0.77$ μm
0.228 ± 0.010	0.230 ± 0.011	0.232 ± 0.011
0.222 ± 0.010	0.228 ± 0.011	0.226 ± 0.011
0.226 ± 0.010	0.237 ± 0.011	0.236 ± 0.011
0.246 ± 0.012	0.246 ± 0.013	0.245 ± 0.013
0.236 ± 0.010	0.240 ± 0.011	0.242 ± 0.011
0.213 ± 0.009	0.216 ± 0.010	0.215 ± 0.010
average value:		
0.229 ± 0.004	0.233 ± 0.005	0.233 ± 0.005

[11] R. PEIERLS: *Proc. Roy. Soc.*, **149** A, 467 (1935).

This formula also follows from Barkas' theory, see equation (16). With the aid of (17) (there is no large difference between (9), (11) and (17) in this region) we derive:

$$B = \frac{1}{W_L - L} \exp\left[-K/(W_L - L)\right] . \tag{19}$$

The constant K was measured on six tracks of beam pions by determination of blob density and gap density. The values of $(W_L - L)^{-1}$ are given in Table III to facilitate derivation of the grain density. The blob density was measured with a 6× eyepiece and a 100× objective. (The obtained value of $(0.183 \pm 3\%)\ \mu m^{-1}$ is about 10% lower than the blob density obtained with a 15× eyepiece and a 100× objective.) The calculated value of K is 1.03 μm. (With a 15× eyepiece K is found to be equal to 0.64 μm.)

4. – Results.

In the Figs. 1*a*, 1*b* and 1*c* the curve for g/g_{min} as a function of the velocity β was calculated. For each β the measured value of g/g_{min} was plotted with the help of relation (3), with three values of parameter A. For the curve with a value $A = 0$, which we adopted as the best, use was made of Table IV.

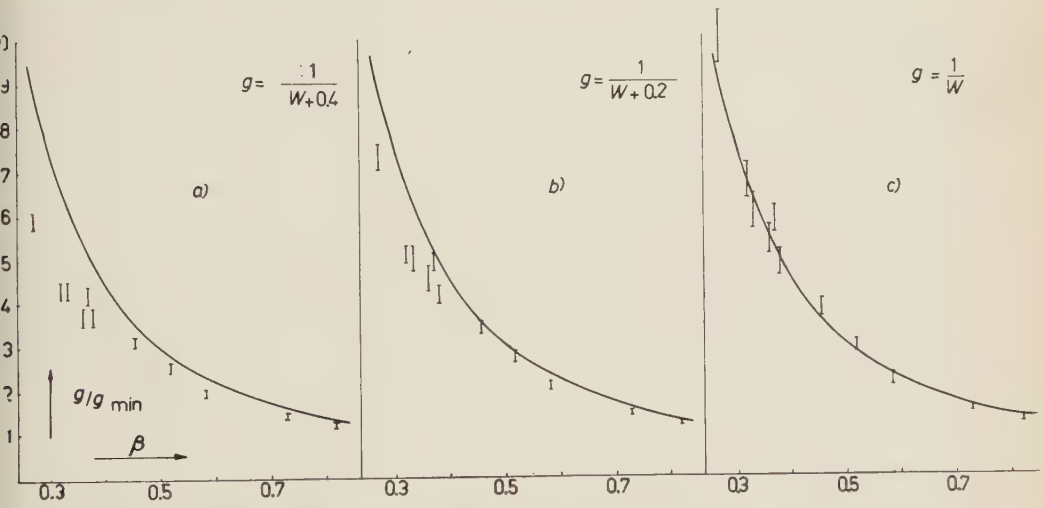

Fig. 1. – g/g_{min} as a function of β as given by formula (1). The measurements are plotted using relation $g = 1/(W + A)$ with $A = 0.4\ \mu m$ in curve 1*a*, $A = 0.2\ \mu m$ in curve 1*b* and $A = 0$ in curve 1*c*. W is an abbreviation of $W_L - L$. At the velocities $\beta = 0.455$ and 0.52 the averaged values of repeated measurements are given (see also Table IV).

TABLE IV. – $1/(W_L - L)$ *as a function of* β *compared with curve* 1c. *The variation of* β *over the interval of measurements is given in column two. Column three shows the calculated values of the relative grain density. These should be compared with the measured values in column four. The last column shows the difference between theory and experiment divided by the statistical error.*

β	Interval of β	g^*_{th}	g^*_{exp}	Q
0.82	0.82 ÷0.82	1.24	1.20	— 0.7
0.73	0.725÷0.735	1.51	1.46	— 0.7
0.585	0.58 ÷0.59	2.24	2.12	— 1.1
0.52	0.51 ÷0.53	2.77	2.90	+ 0.9
0.52	—	—	2.84	+ 0.5
0.455	0.44 ÷0.465	3.54	3.73	+ 1.0
0.455	—	—	3.77	+ 1.2
0.38	0.36 ÷0.40	4.9	4.8	— 0.3
0.37	0.34 ÷0.39	5.2	5.8	+ 2.0
0.36	0.355÷0.365	5.4	5.4	0.0
0.33	0.30 ÷0.36	6.2	6.0	— 0.5
0.32	0.30 ÷0.34	6.7	6.7	0.0
0.27	0.255÷0.30	9.3	10.0	+ 1.2

The measured quantity g of the beam pions gives the trough value g_{min} after multiplying by 0.917, the calculated ratio of the grain density in the trough of the curve as given by JONGEJANS [1] and the grain density of the beam pions.

5. – Conclusion.

The results of the measurements favour the model of Barkas for grain distribution in nuclear emulsion, provided the extrapolation of the relationship between grain density and velocity is correct.

Many authors have reported the mean gap length as a function of the range. For literature see the article of CORTINI *et al.* [12]. It was not possible to make a comparison of the theory presented with the experimental curves given in the literature because reading errors were too large.

* * *

This work is part of the research program of the Institution for Fundamental Research of Matter (F.O.M.), financially supported by the Netherlands Organization for Pure Scientific Research (Z.W.O.).

[12] G. CORTINI, G. LUZZATTO, G. TOMASINI and A. MANFREDINI: *Nuovo Cimento*, **9**, 706 (1958).

The author is indebted to Dr. E. J. LOFGREN and the Bevatron crew for the facilities provided at the Bevatron.

Thanks are due to Professor Dr. G. W. RATHENAU for his helpful criticism, Mr. A. G. TENNER and the members of the group F.O.M. KVII Amsterdam (exp) are thanked for valuable discussions. In particular, the stimulating help of Mr. B. JONGEJANS is gratefully acknowledged. The basic measurements were made by Mrs. F. C. H. HUYSER and by Mrs. H. A. C. BARREVELD. Mr. T. J. VAN DER LINDE performed the greater part of the computations. The calculational aid of Mr. A. D. SASTRADIWIRIA is acknowledged with pleasure.

RIASSUNTO (*)

Questo lavoro descrive le misure di ionizzazione sulle tracce di particelle aventi velocità che corrispondono ad un β tra 0.25 e 0.8. I risultati sono stati ottenuti con la determinazione della lunghezza media di gap. Si calcola una curva teorica con l'aiuto della ben nota relazione fra la densità relativa dei granuli e la velocità, quale è stata verificata da JONGEJANS, e della relazione fra lunghezza media di gap e densità di granuli, data da BARKAS. I risultati sono in accordo con la loro teoria.

(*) *Traduzione a cura della Redazione.*

Tests for $\Delta T = \frac{1}{2}$ in $K^+ \to 3\pi$ and $K_2^0 \to 3\pi$ Decay Modes (*).

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(ricevuto il 9 Giugno 1960)

Summary. — We have studied the consequences of the $\Delta T = \frac{1}{2}$ selection rule on the energy spectra of the pions in $K^+ \to 3\pi$, $K_2^0 \to 3\pi$ decay modes. Without any dynamical assumptions one can conclude that the π^+, π^0 spectra in $K^+ \to \pi^0 + \pi^0 + \pi^+$ and the π^0, π^+ (or π^-) spectra in $K_2^0 \to \pi^+ + \pi^- + \pi^0$ respectively should be the same. The effect of strong final state π-π interactions on the general structure of the decay amplitude is discussed and the possibility of detecting experimentally the presence of such strong interactions is suggested. The question of whether the experimental spectra provide a critical test of $\Delta T = \frac{1}{2}$ rule is also considered.

1. – Introduction.

Several tests of the $\Delta T = \frac{1}{2}$ rule in $K^{\pm}$ and in K^0, $\overline{K}^0$ decays into three pions have been proposed [1-3]. These tests depend upon a combination of symmetry properties of the allowed final isotopic spin states with assumptions about the momentum dependence of the matrix elements. Weinberg recently has suggested a test of $\Delta T = \frac{1}{2}$ based on a comparison of the energy spectra in τ^+ and τ'^+ decays [2], using an expansion of the matrix element in powers of the pion kinetic energies. Treiman and Weinberg have suggested a test based upon time dependent interference effects on the energy spectra in K_1^0 and K_2^0 decays into three pions [3].

(*) Supported in part by the U.S. Atomic Energy Commission under Contract No. AT(11-1)-30 and in part by the Research Committee of the University of Wisconsin with funds provided by the Wisconsin Alumni Research Foundation.

(1) A. Pais: *Proceedings of the Seventh Annual Rochester Conference* (New York, 1957), Section 8, p. 1.

(2) S. Weinberg: *Phys. Rev. Lett.*, **4**, 87 (1960).

(3) S. B. Treiman and S. Weinberg: *Phys. Rev.*, **116**, 239 (1959).

We have investigated the relations implied by the $\Delta T=\frac{1}{2}$ rule between the energy spectra in τ^+, τ'^+ decay and the spectra in K_2^0 decay into three pions. The energy spectra in $\tau'^+\to 2\pi^0+\pi^+$ and in $K_2^0\to\pi^++\pi^-+\pi^0$ are related without dynamical assumptions. To find further relations involving also the τ^+ mode and the mode $K_2^0\to 3\pi^0$ we utilize an expansion similar to Weinberg's, but modified to include terms which would result from strong final state pion-pion interactions and which are not expansible in Weinberg's manner. The detection of these terms would be a test of the presence of strong final state interactions.

Unfortunately these relations between the energy spectra of the various processes will not be critical tests of $\Delta T=\frac{1}{2}$ if the momentum dependence of the matrix element is entirely due to final state interactions. The reason for this is similar to the reason that the branching ratios, τ^+/τ'^+ and $K_2^0\to\pi^++\pi^-+\pi^0/K_2^0\to 3\pi^0$ are not good tests of $\Delta T=\frac{1}{2}$ [4]. In $K^+\to 3\pi$ the only effect of the $\Delta T=\frac{1}{2}$ rule is to require a $T=1$ final state. If the matrix element is a constant it is well known that the rule, $\Delta T=\frac{1}{2},\frac{3}{2}$, restricts the final state to the unique, totally symmetric one of $T=1$. Since the main contribution to the total decay rate is presumably from this symmetric state, the τ/τ' branching ratio is not a good test for the presence of $\Delta T=\frac{3}{2}$. One must therefore look at the deviations from a constant matrix element spectrum to find a test for $\Delta T=\frac{3}{2}$.

Final state pion-pion interaction may be responsible for the observed deviations of the matrix element from a constant [5,6]. Under this assumption we first imagine the final state interaction turned off. The resulting amplitude is all in the symmetric $T=1$ state. When the final state interaction is switched on, transitions to other $T=1$ states will result, but no $T=2$ amplitude will be induced since the pion-pion coupling is charge independent. Hence a comparison of the energy spectra in τ^+ and τ'^+ decays would not be a way of detecting $\Delta T=\frac{3}{2}$ in the fundamental weak interaction if final state effects are the source of these spectra.

The same arguments are valid in the case of K_2^0 decay into three pions. Again the effect of $\Delta T=\frac{1}{2}$ is to pick out the $T=1$ final state, the $T=0$ state being forbidden if CP is conserved. The only critical test of the isotopic selection rule which is valid in every circumstance is the comparison of the $K_2^0\to 3\pi$ and $K^+\to 3\pi$ partial lifetimes [1].

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[4] R. H. Dalitz: *Rev. Mod. Phys.*, **31**, 823 (1959).

[5] R. F. Sawyer and K. C. Wali: preprint; N. N. Khuri and S. B. Treiman: preprint.

[6] B. S. Thomas and W. G. Holladay: *Phys. Rev.*, **115**, 1329 (1958); A. N. Mitra: *Nuclear Physics*, **6**, 404 (1958).

2. – The amplitude for $K \to 3\pi$.

We write a general isotopic vector amplitude for three pions as $\boldsymbol{A}(\omega_1, \omega_2, \omega_3)$ and a scalar amplitude as $B(\omega_1, \omega_2, \omega_3)$, where ω_i are the pion energies. $\Delta T = \frac{1}{2}$ is achieved by coupling components of these states to the K particles with the correct coefficients. We use a notation of an effective Hamiltonian which acts to annihilate the initial particle and create the final ones. K^-, K^+, $\overline{K}^0$, K^0 annihilate the K^-, K^+, $\overline{K}^0$ and K^0 respectively. A^+, A^-, A^0 create three pion $T = 1$ states. B^0 creates the $T = 0$ state.

We write

$$(1) \qquad H'_{\text{eff}} = A^+K^+ + \frac{1}{\sqrt{2}} A^0K^0 + B^0K^0 + \\ + [(CP)A^+][(CP)K^+] + \frac{1}{\sqrt{2}} [(CP)A^0][(CP)K^0] + [(CP)B^0][(CP)K^0] .$$

Here we have assumed CP invariance in the weak interaction. Using the results [7] that

$$(CP)A^0 = -A^0 ,$$

$$(CP)B^0 = \quad B^0 ,$$

we obtain

$$(2) \qquad H'_{\text{eff}} = A^+K^+ + A^0K_2^0 + A^-K^- + \sqrt{2}\, B^0K_1^0 .$$

It is evident that we require only the amplitude $\boldsymbol{A}(\omega_1, \omega_2, \omega_3)$ to discuss K^+ and K_2^0 decays. Bose statistics dictate the following form for $\boldsymbol{A}(\omega_1, \omega_2, \omega_3)$:

$$(3) \qquad \boldsymbol{A}(\omega_1, \omega_2, \omega_3) = (\boldsymbol{\varphi}_1 \cdot \boldsymbol{\varphi}_2)\boldsymbol{\varphi}_3 f(\omega_1, \omega_2, \omega_3) + \\ + (\boldsymbol{\varphi}_2 \cdot \boldsymbol{\varphi}_3)\boldsymbol{\varphi}_1 f(\omega_2, \omega_3, \omega_1) + (\boldsymbol{\varphi}_3 \cdot \boldsymbol{\varphi}_1)\boldsymbol{\varphi}_2 f(\omega_3, \omega_1, \omega_2) ,$$

with $f(x, y, z) = f(y, x, z)$. The $\boldsymbol{\varphi}$'s are isotopic spin vectors for the three pions. From (3) we deduce the following matrix elements for the various decay modes:

$$(4) \qquad \begin{cases} \langle \pi_a^+ \pi_b^+ \pi^- | T | K^+ \rangle = f(\omega_a, \omega_-, \omega_b) + f(\omega_b, \omega_-, \omega_a) \\ \langle \pi_a^0 \pi_b^0 \pi^+ | T | K^+ \rangle = f(\omega_a, \omega_b, \omega^+) , \\ \langle \pi^+ \pi^- \pi^0 | T | K_2^0 \rangle = f(\omega_+, \omega_-, \omega_0) , \\ \langle \pi_a^0, \pi_b^0, \pi_c^0 | T | K_2^0 \rangle = f(\omega_a, \omega_b, \omega_c) + f(\omega_c, \omega_a, \omega_b) + f(\omega_b, \omega_c, \omega_a) . \end{cases}$$

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[7] R. GATTO: *Phys. Rev.*, **106**, 168 (1957); G. SNOW: *Phys. Rev.*, **103**, 1111 (1956).

In the case of the τ^+ and τ'^+ modes ω_a and ω_b stand for the energies of the two identical pions.

We note a close connection between the amplitude for τ'^+ decay and that for $K_2^0\to\pi^+ +\pi^- +\pi^0$. The π^0 energy spectrum in $K_2^0\to\pi^+ +\pi^- +\pi^0$ is identical to the π^+ spectrum in τ'^+ decay and the energy spectrum of π^+ or π^- in $K_2^0\to\pi^+ +\pi^- +\pi^0$ is identical to the π^0 spectrum in τ'^+ decay. To go farther we need dynamical assumptions.

Weinberg's expansion of the amplitude in a series of powers of the squared momenta is not valid in the presence of strong final state interactions. This fact has been noted in reference ([5]) and is connected with the phases induced by final state interactions. In reference ([5]) it is found that the imaginary part of the amplitude cannot be expanded in a series of integral powers of the squared pion momenta. Instead we use a modified expansion suggested by the work of reference ([5]). To make this expansion plausible we compute the imaginary part of the decay amplitude to first order in a final state pion-pion interaction. This interaction will act on an amplitude which is otherwise real to produce an effect on the spectra and to induce phases in the matrix element.

We assume a K-decay amplitude which in the absence of final state interactions is expansible in a power series of the invariants, which we choose in this case as

$$(5)\qquad \begin{cases} v_1 = -\frac{1}{4}(p_2+p_3)^2-\mu^2\,, \\ v_2 = -\frac{1}{4}(p_3+p_1)^2-\mu^2\,, \\ v_3 = -\frac{1}{4}(p_1+p_2)^2-\mu^2\,. \end{cases}$$

p_1, p_2, and p_3 are the respective pion four-momenta. The variable, *e.g.* v_1, is the square of the three-momentum of particle number 2 or 3 in the center of the momentum of particles number 2 and 3. The variable v_i is also expressible in terms of the energy, ω_i, of the i-th pion in the rest frame of the K-particle,

$$v_i = (1/4\mu^2)(M_K^2 - 2M_K\omega_i - 3\mu^2)\,.$$

We take

$$(6)\qquad \langle 3\pi|T|K\rangle = (\omega_1'\omega_2'\omega_3')^{-\frac{1}{2}}F(v_1', v_2', v_3')$$

in the absence of final state interactions. To lowest order in the $\lambda\varphi^4$ theory we have

$$(7)\qquad \langle 3\pi|T^\dagger|3\pi'\rangle = (\omega_2'\omega_3'\omega_2\omega_3)^{-\frac{1}{2}}\,a\,\delta^3(\boldsymbol{p}_1-\boldsymbol{p}_1') + \\ + (\omega_1'\omega_3'\omega_1\omega_3)^{-\frac{1}{2}}\,b\,\delta^3(\boldsymbol{p}_2'-\boldsymbol{p}_2') + (\omega_1'\omega_2'\omega_1\omega_2)^{-\frac{1}{2}}\,c\,\delta^3(\boldsymbol{p}_3-\boldsymbol{p}_3')\,,$$

where a, b, and c are real constants depending upon the coupling constants in the various charge states.

We calculate the imaginary part of the decay amplitude to first order in the pion-pion coupling from the unitarity condition, $2\,\mathrm{Im}\,T = T^\dagger T$. The only intermediate states are those of three pions. A non-relativistic calculation has those features in which we are interested. The unitarity condition gives

$$(8)\qquad \delta(E_K - E_{3\pi})\,\mathrm{Im}\,\langle 3\pi|T|K\rangle = \mathrm{const}\int \mathrm{d}^3p_1' \int \mathrm{d}^3p_2'\,\delta(E_K - E_{3\pi'})\,\delta(E_{3\pi'} - E_{3\pi})\cdot$$
$$\cdot F(\nu_1', \nu_2', \nu_3')\{a\,\delta^3(\boldsymbol{p}_1 - \boldsymbol{p}_1') + b\,\delta^3(\boldsymbol{p}_2 - \boldsymbol{p}_2') + c\,\delta^3(\boldsymbol{p}_3 - \boldsymbol{p}_3')\}\,.$$

When $F(\nu_1', \nu_2', \nu_3')$ is expanded in a power series and the integratiosn are performed we obtain

$$(9)\qquad \mathrm{Im}\,\langle 3\pi|T|K\rangle = \nu_1^{\frac{1}{2}} g_1(\nu_1, \nu_2, \nu_3) + \nu_2^{\frac{1}{2}} g_2(\nu_1, \nu_2, \nu_3) + \nu_3^{\frac{1}{2}} g_3(\nu_1, \nu_2, \nu_3)\,,$$

where the g's are expressed as power series in ν_i and

$$\nu_1 = (1/4\mu^2)(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2\,, \qquad \nu_2 = (1/4\mu^2)(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2\,, \qquad \nu_3 = (1/4\mu^2)(\boldsymbol{p}_1 - \boldsymbol{p}_2)^2\,.$$

The $\nu^{\frac{1}{2}}$ factors cannot be expanded in power series in p_i^2 and can give rise to deviations from Weinberg's predictions [2].

3. – Power series expansion.

In accordance with the above result we propose fitting the $\mathrm{K} \to 3\pi$ data by developing the function $f(\omega_1, \omega_2, \omega_3)$ in (4) in the form (changing the variables to $\nu_i = (1/4\mu^2)(M_K^2 - 2M_K\omega_i - 3\mu^2)$),

$$(10)\qquad f(\nu_1, \nu_2, \nu_3) = \mathrm{const}\,(1 + B\nu_3 + C(\nu_1 + \nu_2) + i[\alpha\sqrt{\nu_3} + \beta(\sqrt{\nu_1} + \sqrt{\nu_2})])\,.$$

This form of expansion also follows from the results of reference [5] where the dependence of $f(\nu_1, \nu_2, \nu_3)$ upon its arguments is attributed to pion-pion interactions. The results of reference [5] can be used to determine our function, $f(\nu_1, \nu_2, \nu_3)$. However, these results depend on assumptions about the form of the relevant dispersion relations for $\mathrm{K} \to 3\pi$ and on assumptions about the behaviour of pion-pion phase shifts. Here we instead shall discuss the possibilities of determining some of the constants in (10) experimentally. In particular if the existence of the $\nu^{\frac{1}{2}}$ terms could be established, then one would have strong evidence for large pion-pion phase shifts. If, on the other hand, the $\nu^{\frac{1}{2}}$ terms do not arise, we shall have a good test for $\Delta T = \frac{1}{2}$.

The energy spectra for the various pions in the various decays have been calculated using (4) and (10). We use the following notation to express the

results for the single pion energy spectra:

$s_-(\omega)$ — distribution of π^- in $K^+\to\pi^+ + \pi^+ + \pi^-$

$s_+(\omega)$ — distribution of π^+ in $K^+\to\pi^+ + \pi^+ + \pi^-$

$s'_+(\omega)$ — distribution of π^+ in $K^+\to\pi^0 + \pi^0 + \pi^+$

$s'_0(\omega)$ — distribution of π^0 in $K^+\to\pi^0 + \pi^0 + \pi^+$

$t_+(\omega) = t_-(\omega)$ — distribution of π^+ or π^- in $K_2^0\to\pi^+ + \pi^- + \pi^0$

$t_0(\omega)$ — distribution of π^0 in $K_2^0\to\pi^+ + \pi^- + \pi^0$

$t_0(\omega)$ — distribution of π^0 in $K_2^0\to\pi^0 + \pi^0 + \pi^0$.

The exact relations following from (4) are

$$\text{(11)}\qquad \begin{cases} t_+(\omega) = 2s'_0(\omega) \\ t_0(\omega) = 2s'_+(\omega) \\ t'_0(\omega) = \frac{1}{3}[2\,s_+(\omega) + s_-(\omega) - 2s'_0(\omega) - s'_+(\omega)]\,. \end{cases}$$

These spectra are normalized to give the relative total decay rates upon integration over the energy.

From our expansion to order v in (10) we may write the four τ^+ and τ'^+ spectra in terms of four constants. The K_2^0 spectra are determined from the τ^+ and τ'^+ spectra by (11). We write the τ and τ' spectra in terms of a factor $D(t)$ which is a constant times the density of states energy spectrum, four constants, and two functions $I(t)$ and $J(t)$ plotted in Fig. 1. We obtain:

$$s_-(t) = \frac{X^2}{\gamma^2}\,D(t)\left[1+\gamma t+\frac{\gamma^2}{X^2}\left\{\frac{.86\,F^2+.61(4G^2-F^2)}{F}\,(1+\lambda t)+2F\,I(t)+8G\,J(t)\right\}\right],$$

$$s_+(t) = \frac{X^2}{\gamma^2}\,D(t)(1+.27\,\gamma)\left[1-\frac{\gamma}{2+.54\,\gamma}\,t+\frac{\gamma^2}{X^2}\,\frac{1}{(1+.27\gamma)}\cdot\right.$$
$$\left.\cdot\left\{\frac{.86\,F^2+.61\,(4\,G^2-F^2)}{F}\,(1+.27\lambda)\left(1-\frac{\lambda}{2+\cdot 54\,\lambda}\,t\right)+4\,GI(t)+2(F+2G)J(t)\right\}\right],$$

$$s'_+(t) = \frac{X^2}{\gamma'^2}\,D(t)\left[1+\gamma'(t)+\frac{\gamma'^2}{4}\,t^2+\frac{\gamma'^2}{X^2}\cdot\right.$$
$$\left.\cdot\left\{\frac{.86\,G^2+.61\,F(F-2G)}{F}(1+\lambda' t)+\frac{2G^2}{F}\,I(t)+\frac{4G(F-G)}{F}\,J(t)\right\}\right].$$

$$s'_0(t) = \frac{X^2}{\gamma'^2}\,D(t)(1+.27\,\gamma')\left[1-\frac{\gamma'}{2+.54\,\gamma'}\,t+\frac{\gamma'^2}{X^2}\,\frac{1}{1+.27\,\gamma'}\cdot\right.$$
$$\left.\cdot\left\{\frac{.86\,G^2+61-F(F-2G)}{F}\,(1+.27\lambda')\left(1-\frac{\lambda'}{2+.54\lambda'}\,t\right)+\frac{2G(F-G)}{F}\,I(t)+2G\,J(t)\right\}\right],$$

where

$$\gamma' = -\frac{2\gamma}{1+.27\gamma}, \qquad \lambda = \frac{1.77(F^2-4G^2)}{.86F^2+.61(4G^2-F^2)},$$

$$\lambda' = \frac{1.77F(2G-F)}{.86G^2+.61F(F-2G)}.$$

G and F are constants which vanish if the imaginary part of the amplitude vanishes. In this case we get a linear approximation similar to Weinberg's and the percentage rise in each spectrum from zero to maximum kinetic energy is given in terms of one parameter, γ, which is measured directly from the π^- spectrum in τ^+-decay. In the case of the π^+ spectrum in τ'^+ decay a quadratic term has been retained. This is necessary because in this one case the linear term in the amplitude will turn out to be of the same order as the constant term. Our linear approximation differs from that of Weinberg by our inclusion in equation (10) of terms of order QB and QC (where Q is the energy release) in the new constant terms resulting from substitutions of the relation, $\nu_1+\nu_2+\nu_3=(\frac{1}{4}\mu^2)Q(M_K+3\mu)$ into (10). Though inclusion of these terms apparently gives an additional term in the linear approximation, it does not in fact. If we normalize the spectra to one at the lower limit, our linear forms are as follows:

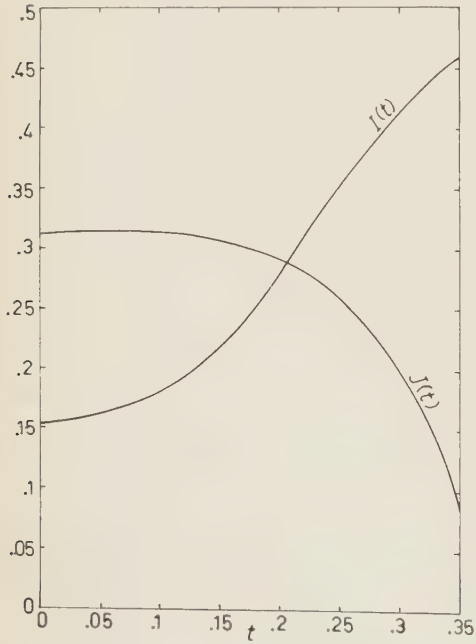

Fig. 1. – The functions $I(t)$ and $J(t)$.

$$(12a) \quad s_-(t) = 1+\gamma t,$$

$$(12b) \quad s_+(t) = 1-\frac{\gamma}{2+0.54\gamma}t,$$

$$(12c) \quad s'_+(t) = 1+\gamma' t+\gamma'^2\frac{t^2}{4},$$

$$(12d) \quad s'_0(t) = 1-\frac{\gamma'}{2+0.54\gamma'}t,$$

where

$$\frac{2}{\gamma'} = -\frac{1}{\gamma}-0.27.$$

The variable t ranges from zero to .35. The best estimate of γ is from reference [8] and is about $\gamma=1.5$. A more refined estimate

[8] S. McKenna, S. Natali, M. O'Connell, J. Tietge and N. C. Varshneya: *Nuovo Cimento*, **10**, 763 (1958).

could be made from equations (12*a*) and (12*b*) by utilizing data on both the π^+ and π^- spectra in τ^+ decay ([8]).

4. – Conclusion.

Without extensive data one could not hope to determine the other three constants which enter when the imaginary terms are present and which give departures from linearity. However it is hoped that one could eventually detect the presence of the contributions involving $I(t)$ and $J(t)$ if not their exact coefficients.

We believe that further data on $K^+\to 3\pi$ decay might provide evidence for or against final state interactions. The best test of $\Delta T=\frac{1}{2}$, however, would be from the exact relations, (11), relating K_2^0 decay into three pions to the τ^+ and τ'^+ modes.

RIASSUNTO (*)

Abbiamo studiato le conseguenze della regola di selezione $\Delta T=\frac{1}{2}$ sugli spettri di energia dei pioni nei modi di decadimento $K^+\to 3\pi$ e $K_2^0\to 3\pi$. Senza nessuna ipotesi dinamica si può concludere che gli spettri π^+, π^0 nella $K^+\to\pi^0+\pi^0+\pi^+$ e gli spettri π^0, π^+ (o π^-) mella $K_2^0\to\pi^+ +\pi^+ +\pi^0$ devono essere gli stessi. Si discute l'effetto delle interazioni forti π-π nello stato finale sulla struttura generale dell'ampiezza di decadimento e si suggerisce la possibilità di scoprire sperimentalmente la presenza di queste interazioni forti. Si esamina anche il problema se gli spettri sperimentali forniscono una riprova critica della regola $\Delta T=\frac{1}{2}$.

(*) *Traduzione a cura della Redazione.*

A Note on the Proof of Dispersion Relations (*).

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(ricevuto il 17 Giugno 1960)

Summary. — Lehmann's proof of dispersion relations is completed by showing that the absorptive part of the scattering amplitude is regular in a uniform neighbourhood of the real axis as a function of the meson mass. The same point is considered for production processes; one or two other difficulties in the latter case are also discussed.

1. – Introduction.

In the usual proof of dispersion relations, according to the technique of Bogoljubov ([1,2]), the most important step is the analytic continuation of the absorptive part A of the amplitude from a non-physical value of the meson mass $\sqrt{\zeta}$ up to the physical value μ. Using the Jost-Lehmann-Dyson representation ([3]), Lehmann ([4]) was able to show that this continuation is possible provided that the momentum transfer Δ is not too large, but may be larger than the values for which dispersion relations had been proved in ([1,2]). But Lehmann's proof is not quite complete, because what is required is that an integral of A over the energy W', namely

$$\int_{(m+\mu)^2}^{\infty} \mathrm{d}W'^2 A(W', \Delta^2, \zeta) \left\{ \frac{1}{W'^2 - W^2} + \frac{1}{W'^2 - 4\Delta^2 - 2(m^2 + \zeta) + W^2} \right\},$$

(*) This remark is due to R. Stora.

([1]) N. N. Bogoliubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields* (New York, 1959).

([2]) H. J. Bremermann, R. Oehme and J. G. Taylor: *Phys. Rev.*, **109**, 2178 (1958).

([3]) F. J. Dyson: *Phys. Rev*, **110**, 1460 (1958).

is regular in a strip S: (with $\zeta = \zeta_1 + i\zeta_2$)

$$-\Delta^2 \leqslant \zeta_1 \leqslant \mu^2 ,$$

$$|\zeta_2| < \delta$$

and for this, $A(W', \Delta, \zeta)$ must be regular in S for each $W' \geqslant m+\mu$, that is δ must be independent of the energy. In his proof (1), LEHMANN showed that $A(W, \Delta, \zeta)$ is regular, for each W, in a strip $S(W)$, and so the integral over any finite range of W is regular in such a strip. This does not mean that an infinite integral over the energy will be regular. As a well known example, the absorptive part $A(\Delta, W)$ is regular in Δ^2 in a large ellipse (1), but we cannot deduce any regularity in Δ^2 for the scattering amplitude from the dispersion relation, because the ellipse shrinks to the physical region as $W \to \infty$.

In this paper we remedy this objection to Lehmann's method by showing that a uniform strip S exists. First we tidy up the proof of the π-$\mathcal{N}$ dispersion relations, and then the general case.

2. – Pion-nucleon relations.

In the region $\zeta_2 = 0$, $\zeta_1 < -\Delta^2$, the forward tube becomes the upper-half ω-plane, and the backward tube the lower-half ω-plane, where $W^2 = 2\omega\sqrt{\Delta^2 + m^2} + 2\Delta^2 + m^2 + \zeta$. Then we can write

$$(1) \qquad T(\zeta, W, \Delta) = \text{bound state terms} + \\ + \frac{1}{\pi}\left(\int_{(m+\mu)^2}^{W_1^2} \mathrm{d}W'^2 + \int_{W_1^2}^{\infty} \mathrm{d}W'^2\right)\left\{\frac{A(W')}{W'^2 - W^2} + \frac{A(W')}{W'^2 - 4\Delta^2 - 2(m^2+\zeta) + W^2}\right\},$$

where W_1 is a suitable energy to be fixed later. The first integral can be shown to be regular in ζ in a suitable strip S by the method of Lehmann, the existence of δ following from continuity, and because the range of the W' integration is finite. It turns out that the maximum Δ^2 for which the continuation can be done is

$$(2) \qquad \Delta^2_{\max} = k_c^2 + \frac{(m_1^2 - \mu^2)(m_2^2 - m^2)}{W^2 - (m_1 - m_2)^2}$$

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(1) H. LEHMANN: *Nuovo Cimento*, **10**, 579 (1958); *Suppl. Nuovo Cimento*, **14**, 153 (1959).

where

$$k_c^2 = \frac{[W^2-(m+\mu)^2][W^2-(m-\mu)^2]}{4W^2},$$

and m_1, m_2 are certain masses at which the spectrum begins. In the π-$\mathcal{N}$ cases the minimum of the r.h.s. of eq. (2) occurs at $W=m+\mu$. In other cases the minimum need not occur at the lower limit of the W' integration, but will always occur at some finite W, say W_0. We will choose W_1 larger than W_0.

In order to tackle the other integral in eq. (1) more simply, we will deliberately lose information, namely, the rotational invariance of the absorptive part, using the form [4]

$$(3)\qquad A(W,\Delta,\zeta) = \frac{1}{8\pi}\iiiint \frac{\mathrm{d}u_{i0}\,u_i\,\mathrm{d}\mu_i\,\mathrm{d}k_i\,\mathrm{d}\alpha\,\mathrm{d}\chi\,\mathrm{d}\theta_i\,\varphi(u_{i0},u,\alpha,\theta_i,W)}{[x_1(\zeta)-k(\zeta)\cos(\theta-\chi)][x_2(\zeta)-k_c(\zeta)\cos(\chi-\alpha)]},$$

where

$$x_i = \left[k_c^2+u_i^2+k_i^2-(u_0+(u^2-\zeta)/2W)^2\right]/2u_i\sin\theta_i\,.$$

We will discuss the two factors in the denominator separately as functions of ζ. We will show that, for a choise of W_1 which depends on Δ,

$$\mathrm{Re}\,[x_1(\zeta)-k_c\cos(\theta-\chi)] \quad \text{and} \quad \mathrm{Re}\,[x_2(\zeta)-k_c\cos(\chi-\alpha)]$$

are always positive over the support of φ, provided that ζ_2 lies in a strip $|\zeta_2|<\delta$.

Consider first the simplest case $\Delta=0$. We know that $x_{1,2}$ is linear in ζ and that

$$(\mathrm{Re}\,x_{1,2})^2 \geqslant \left(\frac{W^2+m^2-\zeta_1}{2W}\right)^2 - m^2 + \frac{(m_1^2-\mu^2)(m_2^2-m^2)}{W^2-(m_1-m_2)^2},$$

and so a sufficient condition that the denominator in eq. (3) does not vanish is

$$(4)\qquad \zeta_2^2 < \left(\frac{4W^2}{W^2+m^2}\right)^2\left[\frac{8\mu^2k_1^2(2m\mu+\mu^2)}{W^2}+\frac{\{8\mu^2(2m\mu+\mu^2)\}^2}{W^2}\right],$$

where

$$k_1^2 = \frac{(W^2+m^2-\zeta_1)^2}{4W^2} - m^2\,.$$

Now because the r.h.s. of eq. (4) has a positive definite minimum when W ranges from W_1 to ∞, and $\zeta_1 \leqslant \mu^2$, it follows that there exists a uniform strip around $\zeta_1 \leqslant \mu^2$ where A is regular in ζ.

In the case $\Delta^2 \neq 0$ we will show that

(*a*) $$\operatorname{Re} x_1 > \operatorname{Max}\left[\left|\operatorname{Re} k_c\left(1-\frac{2\Delta^2}{k^2}\right)\right||\cos\alpha| + \left|\operatorname{Re}\left(\frac{2\Delta}{k_c}(k_c^2-\Delta^2)\right)^{\frac{1}{2}}\right||\sin\alpha|\right],$$

(*b*) $$\operatorname{Re} x_2 > \operatorname{Re} k_c .$$

The ζ dependence is contained in

$$k_c^2 = \frac{[W^2+m^2-(\zeta_1+i\zeta_2)]^2}{4W^2} - m^2 = k_1^2 + ik_2^2 , \quad \text{say} .$$

It can be shown by elementary majorizations that sufficient for condition (*a*) is

(5) $$\mu^2(2m\mu+\mu^2) > \frac{1}{4}K_2^2 + Q\left(\frac{1}{k_1^2}\right),$$

where $Q(1/k_1^2)$ is a polynomial in k_1^{-2} whose coefficients are polynomials in Δ^2 and ζ_2. Provided that k_1 is sufficiently big, condition (5) leads to condition (4), which is the same as condition (*b*) as well. Hence, by choosing W in eq. (1) large enough, *i.e.* so that eq. (5) is satisfied, the existence of a uniform strip S is proved.

3. – Production processes.

Dispersion relations for the process $a+b \to c+d$ have been proved [5] by the method of Lehmann for a certain range of Δ^2 and the masses, which includes [6] the process $\Lambda+\pi \to \Sigma+\pi$, but not associated production. For the proof, analytic continuation in the masses $m_b^2 = \zeta$ and $m_d^2 = \beta$ must be performed. It is preferable [7] to keep $\zeta_1 - \beta_1$ equal to its physical value $m_b^2 - m_d^2$, and to regard $(\zeta+\beta)/2 = \zeta'$ as the variable in which the continuation is made. The forward tube (R) reduces to the form before eq. (3.1) of reference [5] only if $\zeta_1 = \beta_1$. But also without this simplification, (R) is the upper half plane if ζ' is sufficiently negative. As we continue in ζ', a branch point in the momentum $K_1(\zeta')$ occurs at some value, if the energy is below threshold. This was not treated completely in [5,6]. However, the branch gives no trouble, since we can cut the ζ'-plane from the relevant value of ζ', the cut lying everywhere below the real axis. Then a path of continuation

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[5] R. F. Streater: *Nuovo Cimento*, **13**, 57 (1959).
[6] Y. S. Jin: *Nuclear Physics*, **15**, 102 (1960).
[7] R. Oehme and J. G. Taylor: *Phys. Rev.*, **103**, 371 (1959).

in ζ' from large negative values up to the physical value can be chosen, with $\zeta'_2 > 0$, lying in the forward tube (R). Just as above we can prove regularity in a strip S (except for the cut) and so dispersion relations are proved for a limited range of Δ^2 and the masses. If the initial and final particles have not got the same mass, $\Delta^2 < 0$ is a physical possibility for some energies. But a dispersion relation with $\Delta^2 < 0$, if it could be proved, would have an infinite unphysical region. This is because the smallest physical value of Δ^2 tends to zero as the energy tends to infinity (for then the mass-differences can be neglected, and we tend to the equal-particle case). If $\Delta^2 < 0$ the exponential $\exp[i(k+k')\cdot x]$ is rapidly increasing for some x in the forward cone for any complex ω, and so dispersion relations cannot be proved (*).

* * *

The second named author would like to thank Drs. R. Omnès, M. Froissart and R. Stora for many helpful discussions.

(*) This remark is due to R. Stora.

RIASSUNTO

La dimostrazione di Lehmann delle relazioni di dispersione è completata dimostrando che la parte assorbente dell'ampiezza di scattering è regolare in un intorno uniforme dell'asse reale come funzione della massa del mesone. Si discutono inoltre alcune questioni connesse colle relazioni di dispersione per processi di produzione.

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

On the Gauge Properties of Green's Functions.

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(ricevuto il 18 Giugno 1960)

Summary. — The transformation properties of Green's functions in quantum electrodynamics are derived by functional analysis. Also, the generalized Ward identities and branching equations are obtained from differential equations for the generating functional.

The transformation properties of Green's functions under a gauge transformation were derived recently by Okubo [1] with the use of the formalism worked out by Caianiello [2]. The purpose of the present paper is to show that these results can be derived in a simple manner using the formalism of functional differentiation [3]. Caianiello's branching equations [2] also follow from a functional analysis in a trivial fashion.

One can derive the transformation properties of Green's functions from the transformation law for the generating functional. The generating functional $Z\{\eta\bar{\eta}J_\mu\}$ for unnormalized Green's functions $G_{\mu_1\dots\mu k}(x_1 \dots x_n,\ y_1 \dots y_n,\ z_1 \dots z_k)$ is simply the vacuum expectation value of the S operator in the presence of external sources,

$$(1) \qquad Z\{\eta\bar{\eta}J_\mu\} = \langle S\rangle_0 = \left\langle T \exp i\int \mathscr{L}(x)\,\mathrm{d}x \right\rangle =$$

$$= \left\langle T \exp i\int (j^\mu A_\mu + \bar{\eta}\psi + \bar{\psi}\eta + J^\mu A_\mu)\,\mathrm{d}x \right\rangle ,$$

(*) On leave of absence from Warsaw University, Warsaw.

(1) S. Okubo: *Nuovo Cimento*, **15**, 949 (1960).

(2) E. R. Caianiello: *Nuovo Cimento*, **13**, 640 (1959).

(3) The functional analysis was also used by B. Zumino «*Journ. Math. Phys.*, **1**, 1 (1960)» to study the problems of gauge invariance. However, his approach is different from ours.

$$(2)\qquad G_{\mu_1\dots\mu_k}(w_1\dots x_n,\ y_1\dots y_n,\ z_1\dots z_k) = \\ = i^{-2n-k}\,\frac{\delta^{2n+k}}{\delta\eta(y_1)\dots\delta\eta(y_n)\,\delta\bar\eta(x_1)\dots\delta\bar\eta(x_n)\,\delta J^{\mu_1}_{(z_1)}\dots\delta J^{\mu_k}_{(z_k)})}\,Z\Big|_{\substack{\eta=0=\bar\eta\\ J_\mu=0}}.$$

The differential equations for the Z functional can be obtained, most easily, with the Wick theorem for the vacuum expectation values [4],

$$(3)\qquad \begin{cases} \langle T A_\mu(z) S\rangle = i\int \mathrm{d}y\ \ T\overbrace{A_\mu(z)\mathscr{L}}(y) S\rangle = \int D^F_{\mu\nu}(x-y)\,\mathrm{d}y\,\langle T(j^\nu(y)-J^\nu(y))\rangle S, \\ \langle T\psi(x) S\rangle = i\int \mathrm{d}y\,\langle T\overbrace{\psi(x)\mathscr{L}}(y) S\rangle = \\ \qquad\qquad = \int S_F(x-y)\,\mathrm{d}y\langle T(e\boldsymbol{A}(y)\psi(y)+\eta(y)) S\rangle, \\ \langle T\bar\psi(x) S\rangle = i\int \mathrm{d}y\,\langle T\overbrace{\mathscr{L}(y)\bar\psi}(x) S\rangle = \\ \qquad\qquad = \int \langle T(e\bar\psi(y)\boldsymbol{A}(y)+\bar\eta(y)) S\rangle\,\mathrm{d}y\,S_F(y-x). \end{cases}$$

Using generalized current conservation in the presence of external sources,

$$(4)\qquad \partial_\mu\langle T(J^\mu(x)+j^\mu(x)) S\rangle = \langle T(J^\mu{}_{,\mu}(x) + ie\bar\psi(x)\eta(x) - ie\bar\eta(x)\psi(x)) S\rangle = \\ = \left(J^\mu{}_{,\mu}(x) + e\frac{\delta}{\delta\eta(x)}\eta(x) - e\bar\eta(x)\frac{\delta}{\delta\bar\eta(x)}\right) Z \overset{\mathrm{df}}{=} \Omega(x) Z,$$

we can get from (3) the following set of equations,

$$(5)\qquad \begin{cases} \left[\Box\frac{1}{i}\frac{\delta}{\delta J^\mu(z)} + J_\mu(z) + e\frac{1}{i}\frac{\delta}{\delta\eta(z)}\gamma_\mu\frac{1}{i}\frac{\delta}{\delta\bar\eta(z)} - \partial_\mu\int\Box M(z-y)\,\mathrm{d}y\,\Omega(y)\right] Z = 0, \\ \left[\left(i\partial - m + e\gamma^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\frac{1}{i}\frac{\delta}{\delta\bar\eta(x)} + \eta(x)\right] Z = 0, \\ \left[\frac{1}{i}\frac{\delta}{\delta\eta(x)}\left(-i\partial - m + e\gamma^\mu\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right) + \bar\eta(x)\right] Z = 0, \end{cases}$$

where $M(x-y)$ gives the gauge dependent part of the photon propagator,

$$(6)\qquad \frac{1}{i}D^F_{\mu\nu}(x-y) = \overbrace{A_\mu(x)A}_\nu(y) = \frac{1}{i}\left(-g_{\mu\nu}D_F(x-y) + \partial_\mu\partial_\nu M(x-y)\right).$$

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[4] N. N. Bogoljubov and D. V. Shirkov: *Introduction to the Theory of Quantized Fields* (New York, 1959), p. 421.

By a gauge transformation we mean, in this paper, an arbitrary change of the function $M(x-y)$ in (6).

The generating functional in a different gauge, $\tilde{Z}$, obeys the set of equations (5) with the M function replaced by a new gauge function, say $\tilde{M}(x-y)$. These two functionals, Z and $\tilde{Z}$, are related to each other by a unitary transformation U,

$$Z = U\tilde{Z} = \exp\frac{i}{2}\int \Omega(x)\, h(x-y)\, \Omega(y)\, \mathrm{d}x\, \mathrm{d}y \cdot \tilde{Z}\,, \tag{7}$$

where $h(x-y) = \tilde{M}(x-y) - M(x-y)$. One can prove this relation using the transformation properties of the functional derivatives under the transformation U,

$$\begin{cases} U^{-1}\dfrac{1}{i}\dfrac{\delta}{\delta J^{\mu}(x)}\, U = \dfrac{1}{i}\dfrac{\delta}{\delta J^{\mu}(x)} - \partial_{\mu}\displaystyle\int h(x-y)\, \mathrm{d}y\, \Omega(y)\,, \\ U^{-1}\dfrac{1}{i}\dfrac{\delta}{\delta \overline{\eta}(x)}\, U = \exp\left[-i\displaystyle\int h(x-y)\, \mathrm{d}y\, \Omega(y)\right]\cdot\dfrac{1}{i}\dfrac{\delta}{\delta\overline{\eta}(x)}\,. \end{cases} \tag{8}$$

Having established the connection between Z and $\tilde{Z}$, one can find the transformation properties of Green's functions. Every term of the expansion of Z (or $\tilde{Z}$) in terms of external spinor sources,

$$Z = \sum_{n=0}^{\infty}\frac{1}{(n!)^2}\int \mathrm{d}x\, \mathrm{d}y\, \overline{\eta}(x_1)\ldots\overline{\eta}(x_n)\, G(x_1\ldots x_n,\, y_1\ldots y_n\,|\,J_{\mu})\, \eta(y_1)\ldots\eta(y_n)\,, \tag{9}$$

is an eigenfunction of the operator U because of the relation

$$\left[\frac{\delta}{\delta\eta(x)} - \overline{\eta}(x)\frac{\delta}{\delta\overline{\eta}(x)}\right]\overline{\eta}(x_1)\ldots\overline{\eta}(x_n)\,\eta(y_1)\ldots\eta(y_n) = \\ = \lambda_n(x)\cdot\overline{\eta}(x_1)\ldots\overline{\eta}(x_n)\,\eta(y_1)\ldots\eta(y_n)\,, \tag{10}$$

where

$$\lambda_n(x) = \sum_{1}^{n}(-)^{n-i}\left(\delta(x-x_i) - \delta(x-y_i)\right).$$

One can, therefore, equate both sides of the eq. (7) term by term,

$$G(x_1\ldots x_n,\, y_1\ldots y_n\,|\,J_{\mu}) = \\ = \exp\left[\frac{i}{2}\int \mathrm{d}x\, \mathrm{d}y\,\left(J^{\mu}{}_{,\mu}(x) + e\lambda_n(x)\right) h(x-y)\left(J^{\mu}{}_{,\mu}(y) + e\lambda_n(y)\right)\cdot\tilde{G}(x_1\ldots x_n,\, y_1\ldots y_n\,|\,J_{\mu})\right]. \tag{11}$$

Derivatives of eq. (11) with respect to J_{μ} at the point $J_{\mu} = 0$, give the transformation formula for an arbitrary propagator. Differentiating, for example

eq. (11) with respect to $J_\mu(z)$, for $n = 1$, one can get the transformation law for the three-field Green function,

$$(12)\quad G_\mu(x, y, z) = \\ = \exp\left[ie^2(h(x-y) - h(0))\right]\left[\tilde{G}_\mu(x, y, z) - i\partial_\mu(f(z-x) - f(z-y))\,\tilde{G}(x, y)\right].$$

This equation was given by Okubo [1] and, for infinitesimal gauge transformation, by Zumino [3].

The generating equation for the set of the Ward identities can be obtained by taking the divergence of the first equation (5),

$$(13)\quad \Box\,\partial^\mu \frac{1}{i}\frac{\delta}{\delta J^\mu(z)} Z = \int \Box^2 f(z-y)\,\mathrm{d}y\,\Omega(y)\,Z\,,$$

where $\Box^2 f(z) = \partial^\mu\partial^\nu D^F_{\mu\nu}(z)$. The function $f(z)$ gives the difference between the propagator $D^F_{\mu\nu}(z)$ and the transverse photon propagator,

$$(14)\quad D^F_{\mu\nu}(z) = (-g_{\mu\nu} + \partial_\mu\partial_\nu\Box^{-1})D_F(z) + \partial_\mu\partial_\nu f(z)\,.$$

The simplest equation which can be derived from eq. (13) is the usual Ward identity

$$(15)\quad \Box_z\,\partial^\mu_z G_\mu(x, y, z) = e(\Box^2 f(z-y) - \Box^2 f(z-x))\,G(x, y)\,.$$

This relation depends on the gauge function $f(z)$. However, the generalized Ward identity, in the form given by Takahashi [5], is valid in all gauges. This can be seen from eq. (15), if the Green function $G_\mu(x, y, z)$ is expressed in terms of the vertex part,

$$(16)\quad G_\mu(x, y, z) = -e\int S(x-\xi)\,\Gamma^\nu(\xi, \eta, \zeta)\,S(\eta - y)\,D_{\mu\nu}(z-\zeta)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta\,.$$

From eqs. (14), (15) and (16) we get

$$(17)\quad \int \Box^2 f(z-\zeta)\,S(x-\xi)\frac{\partial}{\partial\zeta_\nu}\Gamma^\nu(\xi, \eta, \zeta)\,S(\eta - y)\,\mathrm{d}\xi\,\mathrm{d}\eta\,\mathrm{d}\zeta = \\ = (\Box^2 f(z-y) - \Box^2 f(z-x))\,S(x-y)\,,$$

and finally

$$(18)\quad (p_\mu - q_\mu)\Gamma^\mu(p, q) = S^{-1}(p) - S^{-1}(q)\,.$$

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[5] Y. Takahashi: *Nuovo Cimento*, **6**, 371 (1957).

In conclusion we want to show that the branching equations derived by CAIANIELLO (2) by means of combinatorial methods can be easily obtained with the use of the functional technique. The branching equations among kernels (6) are given by functional derivatives of eqs. (4). Caianiello's branching equations with derivatives with respect to mass and charge can be obtained by differentiating the following equations,

$$(19)\qquad \left\{\begin{array}{l} \dfrac{1}{i}\dfrac{\partial Z}{\partial e} = \displaystyle\int \mathrm{d}x \dfrac{1}{i}\dfrac{\delta}{\delta\eta(x)}\gamma^{\mu}\dfrac{1}{i}\dfrac{\delta}{\delta\overline{\eta}(x)}\dfrac{1}{i}\dfrac{\delta}{\delta J^{\mu}(x)} Z\,, \\ \dfrac{1}{i}\dfrac{\partial Z}{\partial m} = \displaystyle\int \mathrm{d}x \dfrac{1}{i}\dfrac{\delta}{\delta\eta(x)}\dfrac{1}{i}\dfrac{\delta}{\delta\overline{\eta}(x)} Z\,. \end{array}\right.$$

Eqs. (19) were derived earlier by HORI (6). They also follow immediately from the Fourier integral form of the Z functional (7),

$$(20)\quad Z = \int \delta\pi\,\delta\chi\,\delta\overline{\chi}\,\exp i\int \mathrm{d}x\left(\tfrac{1}{2}\pi_{\mu}\Box\pi^{\mu} + \overline{\chi}(i\partial\!\!\!/ - m)\chi + e\overline{\chi}\gamma^{\mu}\chi\pi_{\mu} + \overline{\chi}\eta + \overline{\eta}\chi + \pi^{\mu}J_{\mu}\right).$$

* * *

The author wishes to thank Drs. C. J. GOEBEL, J. C. TAYLOR and E. C. G. SUDARSHAN for many stimulating discussions.

(6) Eqs. (20)–(25) of the reference (2).

(7) K. SYMANZIK: *Zeits. f. Naturf.*, **9a**, 809 (1954).

RIASSUNTO (*)

Si derivano con l'analisi funzionale le proprietà di trasformazione delle funzioni di Green nell'elettrodinamica quantistica. Anche le identità generalizzate di Ward e le equazioni di branching vengono ottenute dalle equazioni differenziali per il funzionale generatore.

(*) *Traduzione a cura della Redazione.*

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

Some Properties of Single Loop Diagrams in Perturbation Theory (*).

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(ricevuto il 18 Giugno 1960)

Summary. — Some analytic properties of single loop diagrams are discussed, and in particular it is shown that it is not possible to set up a two-variable spectral representation for production processes.

1. – Introduction.

Methods have recently been developed by various authors [1-3] for examining the analytic properties of a term in the S-matrix corresponding to a given Feynman diagram in perturbation theory. In this paper we shall confine ourselves mainly to diagrams with a single internal closed loop.

A singularity for a given diagram D which does not appear for any of the simpler diagrams obtained by contracting out internal lines of D is termed a « leading » singularity for D. In Section **2** we summarize methods for determining leading singularities, and apply these methods in Section **3** to a discussion of the single-loop vertex function.

(*) Supported in part by United States Air Force, Research and Development Command, Europe.

(1) J. C. Polkinghorne and G. R. Screaton: *Nuovo Cimento*, **15**, 289 and 925 (1960).

(2) L. D. Landau: *J.E.T.P.*, **37** (**10**), 45 (1960); J. C. Taylor: *Phys. Rev.*, **117**, 261 (1960).

(3) J. D. Bjorken: *Spectral Representations of Green's Functions in Perturbation Theory* (preprint).

In Section **4** we consider the four-point function and, using the methods of Tarski ([4]), confirm that no complex singularities appear on the physical sheet when the internal stability conditions are relaxed. In Section **5** our results for the vertex function are used to show that the S-matrix for production processes has only single-variable spectral representations. Pion production in π-$\mathcal{N}$ scattering is further considered in Section **6** and it is shown that there are no anomalous thresholds corresponding to the leading singularities for the various single-loop diagrams.

2. – General methods.

The integral corresponding to a given Feynman diagram is of the form

$$f(z) = \int \dots \int_0^1 \mathrm{d}\alpha_1 \dots \mathrm{d}\alpha_n \frac{\varphi(\alpha)\,\delta(\sum \alpha_i - 1)}{[F(\alpha, z_i)]^\varrho},$$

where the z_i are scalar products of the external momenta together with the squares of the masses on internal lines. It has been shown ([1]) that f has a singularity on most of its sheets when there exists a set of momenta q_i corresponding to the internal lines satisfying the following conditions:

(1) Momentum is conserved at each vertex of the diagram,

(2) $\sum_{\text{loop}} \alpha_i q_i = 0$, the sum being taken round any closed loop of the diagram,

(3) Either $\alpha_i = 0$ or $\dfrac{\partial F}{\partial \alpha_i} = 0$ for each i.

A singularity for which $\partial F/\partial \alpha_i = 0$ for all i is a leading singularity. The lower order singularities, for which some $\alpha_i = 0$, are leading singularities for lower order diagrams obtained by contracting out the corresponding internal lines. The condition $\partial F/\partial \alpha_i = 0$ is equivalent to $q_i^2 = m_i^2$, where m_i is the mass for the i-th line.

Here we shall be concerned principally with diagrams containing only one internal closed loop, and it will be convenient to use the notation proposed by KARPLUS, SOMMERFIELD and WICHMANN ([5]). Denote by p_{ij} the external

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([4]) J. TARSKI: *Journ. Math. Phys.*, **1**, 149 (1960).

([5]) R. KARPLUS, C. M. SOMMERFIELD and E. H. WICHMANN: *Phys. Rev.*, **114**, 376 (1959).

momentum connected to the i-th and j-th internal lines, and generalize for $i-j>1$ by

$$p_{ij}=\sum_{k=j}^{i-1} p_{k\,k+1}\,, \qquad p_{ij}=p_{ji}\,. \tag{4}$$

We define y_{ij} by

$$p_{ij}^2=m_i^2+m_j^2-2m_im_jy_{ij}\,, \qquad y_{ii}=1 \tag{5}$$

and F becomes

$$F=\sum_{i,j}\alpha_i\alpha_jy_{ij}\,, \tag{6}$$

where the α_i have been replaced by $m_i\alpha_i/\sum_j m_j\alpha_j$.

The condition (3) for a leading singularity are, for each i,

$$\sum_j \alpha_jy_{ij}=0$$

and these equations are consistent if

$$\Delta=\det y_{ij}=0\,. \tag{7}$$

We may find the corresponding values of the α_i. If we vary the y_{ij} in such a way that (3) and (7) continue to hold then

$$0=\mathrm{d}F=\sum_{i,j}\alpha_i\alpha_j\,\mathrm{d}y_{ij} \tag{8}$$

and also

$$0=\mathrm{d}\Delta=\sum_{i,j}\Delta_j^i\,\mathrm{d}y_{ij}\,, \tag{9}$$

where Δ_i^j is the cofactor of y_{ij} in Δ. Hence

$$\frac{\alpha_i\alpha_j}{\alpha_k\alpha_l}=\frac{\Delta_j^i}{\Delta_l^k}$$

and in particular

$$\frac{\alpha_i}{\alpha_j}=\frac{\Delta_i^k}{\Delta_j^k} \tag{10}$$

for any k. Thus all the α_i are positive if, for fixed k, Δ_i^k are of the same sign for all i. This is used to decide whether or not the singularity lies on the physical sheet.

3. – The vertex function.

For the single loop diagram with three external lines there are three invariants y_{ij}. We fix y_{13} and impose the stability condition for the external mass:

$$y_{13} > -1 . \tag{11}$$

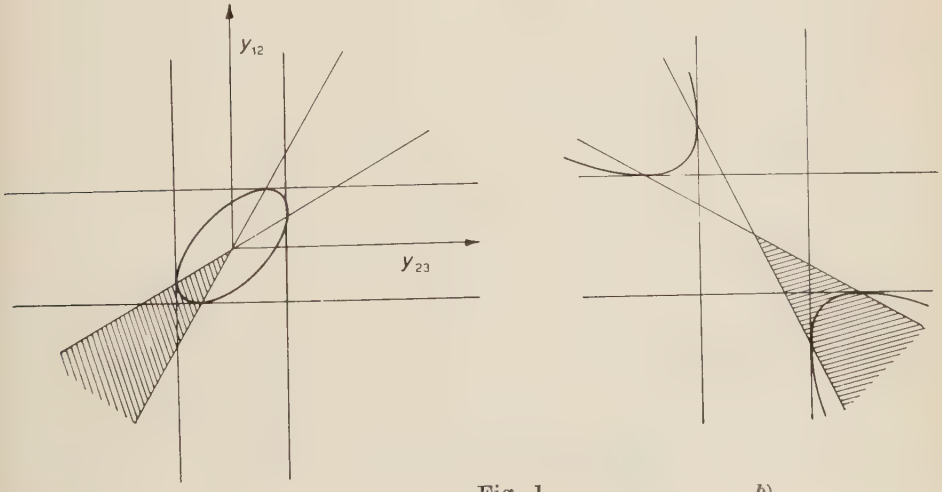
By inspection this implies that F does not vanish for sufficiently large y_{12}, y_{23} and positive α_i. In this region of (y_{12}, y_{23})-space the physical value of the Feynman integral f is obtained by simply integrating along the real axes in the α-planes from 0 to 1, and we may continue the function analytically out of this region by distorting the contours of the multiple integration.

We use the method of Tarski [5] to find the singularities of f on the physical sheet.

If there is internal stability, so that $y_{13} < 1$, the section of $\Delta = 0$ in the real (y_{12}, y_{23})-plane is an ellipse, while if $y_{13} > 1$ it is a hyperbola. In either case the lower order singularities $y_{12} = \pm 1$, $y_{23} = \pm 1$ are tangents to the conic. Of these, only $y_{12} = -1$, $y_{23} = -1$ correspond to positive values of the α and are singular on the physical sheet.

Using equation (10) we find

$$\frac{\alpha_1}{y_{12} - y_{23} y_{13}} = \frac{\alpha_2}{y_{13}^2 - 1} = \frac{\alpha_3}{y_{23} - y_{13} y_{12}} . \tag{12}$$



a) Fig. 1. b)

It follows that the part of the conic having positive α's lies in the shaded region of Fig. 1 and is the arc of the curve between the points of tangency

with $y_{12} = -1$, $y_{23} = -1$. Applying the argument used by Tarski for the four-point function we see that only when $y_{13} < 1$, so that the conic is an ellipse, is there a surface of complex singularities joined to the arc having positive α's.

4. – The four-point function.

If we consider the intersection of horizontal lines with the conics in Fig. 1 we see that, when the vertex function is regarded as a function of one variable y_{12}, with y_{13}, y_{23} fixed

i) if $y_{13}, y_{23} > 1$ there are two real singularities, neither of which is on the physical sheet, in the region $y_{12} > 1$;

ii) if $|y_{13}| < 1$, $y_{23} > 1$ the singularities are complex and are not on the physical sheet.

This enables us to extend Tarski's work on the four-point function to the case when internal stability is relaxed at some or all of the vertices. This is essential for the validity of the Mandelstam representation, since most complicated Feynman graphs have contractions to single loop diagrams for which internal stability does not hold.

The possible forms of the real section of the curve of singularities are shown in Fig. 2. The lower order singularities corresponding to a single contraction in the Feynman diagram, that is a contraction into a vertex part, are again tangents to the curve (when they are real). The asymptotes are the lines $y_{13} = \pm 1, y_{24} = \pm 1$ which are themselves single contractions of the vertex diagrams, and once again $y_{13} = -1$, $y_{24} = -1$ are always singular on the physical sheet. Of the tangents, only those immediately adjacent to these two singular asymptotes can be singular on the physical sheet, but they may not be singular.

To apply the method of Tarski we must decide which parts of the curve have positive α's. Although we could again use equation (10) to do this, the algebra involved is unattractive. Instead, therefore, we note that a part of the curve with positive α's changes to a part at which the α's are not all positive if and only if it has contact with a lower order singularity which itself has positive α's. Further, a lower order singularity which does not have positive α's cannot touch a part of the curve which does have positive α's. This enables us to decide unambiguously which parts of the curve in fact do have positive α's, except in the case of Fig. 2*b*. But in this case it follows quite simply from (10) that it is the arc in the region $y_{13}, y_{24} < -1$ which has positive α's.

It may then easily be confirmed that there are no complex singularities on the physical sheet.

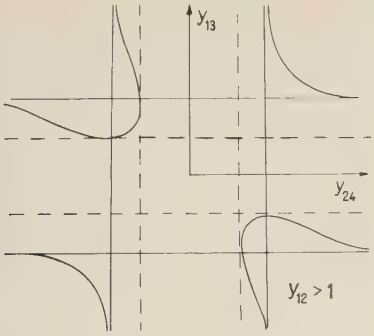


Fig. 2 *a*).

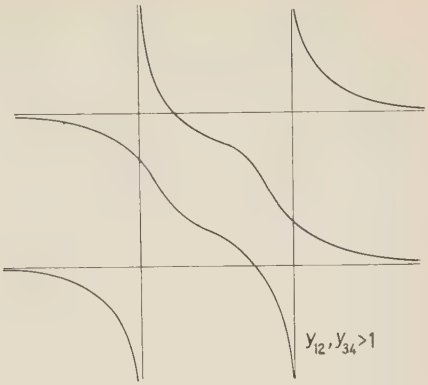


Fig. 2 *b*).

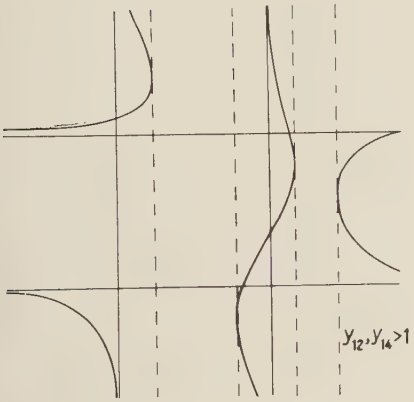


Fig. 2 *c*).

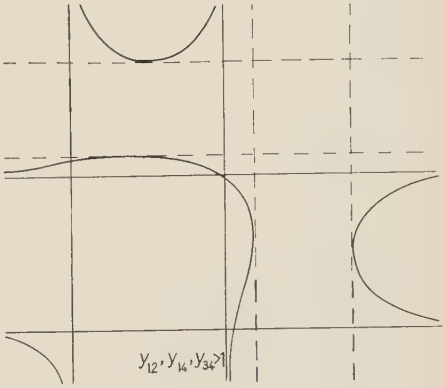


Fig. 2 *d*).

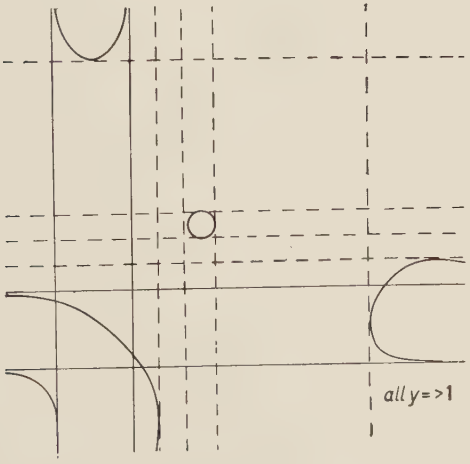


Fig. 2 *e*).

5. – Production processes.

For the five-point function we may take as our invariants the five independent quantities p_{13}^2, p_{24}^2, p_{35}^2, p_{41}^2, p_{52}^2, together with the external masses.

If we fix p_{13}^2, p_{52}^2 the lower-order singularity corresponding to $\alpha_3 = \alpha_5 = 0$ (Fig. 3) is equivalent to a leading singularity for a vertex part with external momenta p_{12}, p_{24}, p_{41}. If the internal stability condition held at the p_{12} vertex of the original diagram it also holds for the contracted diagram. Thus from Section 3, there are complex singularities on the physical sheet (*).

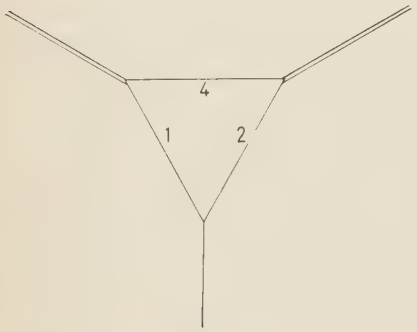


Fig. 3.

If we also fix p_{35}^2 we again have complex singularities corresponding to $\alpha_3 = \alpha_5 = 0$, but if instead we fix p_{24}^2 we may verify that all vertex contractions have two of their external momenta fixed and real, so that no complex singularities can arise from them. It follows that we could only have a double spectral representation for the fifth order single loop diagram if we choose the three fixed invariants to be adjacent, *i.e.*, of the form $p_{i,i+2}$, $p_{i+1,i+3}$, $p_{i+2,i+4}$. This requirement is very restrictive, because if it is satisfied for a given diagram it will no longer be satisfied if we permute the internal lines so that the vertices are connected in a different order. The number of possible permutations will be restricted by conservation laws, but even if there are two such laws in operation there are enough permutations to make impossible a choice of invariants adjacent for them all.

We conclude that the S-matrix for particle-production has only single-variable spectral representations of the usual type. This holds also for processes in which more than one particle is produced.

6. – Pion production in $\pi\mathcal{N}$ scattering.

Single variable dispersion relations for the process

$$\pi + \mathcal{N} \to 2\pi + \mathcal{N}$$

(*) We are indebted to Dr. J. C. POLKINGHORNE for pointing this out.

have been given by SCREATON (6). We here show that the leading singularities for the single loop diagrams do not yield anomalous thresholds for the integration, if all the invariants except the incident energy are fixed at physical values.

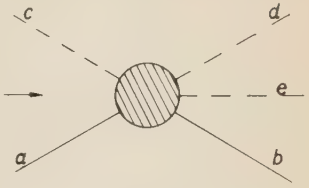


Fig. 4.

There are six allowed ways of connecting the external lines in Fig. 4 to give a single loop diagram (Fig. 5). In each case, if we fix $(c+d)^2$, $(d+e)^2$ $(b+e)^2$, $(a+b)^2$ at physical values and allow $(a+c)^2$ to vary, then only p_{13}^2 or p_{35}^2 or both can vary, p_{52}^2, p_{41}^2 and p_{24}^2 always being independent of $(a+c)^2$.

For the $J=0$ singularity to give an anomalous threshold, from (7) we must have, in particular, for Δ_i^2 the same sign for all i, and similarly for Δ_1^1. However, this condition is incompatible with

$$\sum_i y_{2i}\Delta_i^2 = \sum_i y_{4i}\Delta_i^4 = 0 \,. \tag{13}$$

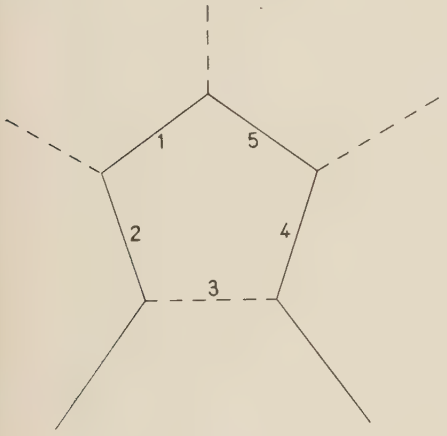


Fig. 5.

This is because the mass variables

$$y_{23} = y_{34} = \mu/2M$$

and

$$y_{13} = 1 - \mu^2/2M^2$$

are positive, as are also y_{24} (which always corresponds to momentum transfer) and at least one of y_{25}, y_{14} (since one and only one of the external mesons is ingoing). Hence either y_{2i} or y_{4i} or both are positive for all i.

* * *

We are grateful to Dr. J. C. POLKINGHORNE for help and encouragement, and to the D.S.I.R. for a grant to each of us.

(6) G. R. SCREATON: *Nuovo Cimento*, **11**, 229 (1959).

RIASSUNTO (*)

Si discutono alcune proprietà analitiche dei diagrammi a nodi semplici, ed in particolare si dimostra che non è possibile costruire una rappresentazione spettrale a due variabili per i processi di produzione.

(*) *Traduzione a cura della Redazione.*

Further Discussion of Possible Experimental Tests for the Paradox of Einstein, Podolsky and Rosen.

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(ricevuto il 20 Giugno 1960)

Summary. — In a previous article we have suggested the experiment of Wu-Shaknov on the annihilation radiation of positronium as a test for the paradox of Einstein, Podolsky and Rosen. In this article, we answer certain criticisms of our conclusions, raised by Peres and Singer. These criticisms are shown to be erroneous, being based on an incorrect interpretation of the polarization of electromagnetic radiation in the quantum domain.

1. – Introduction.

In a previous paper (1), we have discussed the paradox of Einstein, Podolsky and Rosen (2,3), and we have shown that the Wu-Shaknov experiment (4) on the polarization of the annihilation radiation of positronium provides an experimental confirmation of the features of the quantum mechanisms which are at the basis of the above paradox. In a recent article, Peres and Singer (5) criticize our conclusions and state that this experiment does not actually provide such a confirmation. We shall show with the aid of a more detailed analysis of the meaning of polarization of photons in quantum mechan-

(1) D. Bohm and Y. Aharonov: *Phys. Rev.*, **108**, 1070 (1957).
(2) A. Einstein, B. Podolski and N. Rosen: *Phys. Rev.*, **47**, 777 (1935).
(3) W. H. Furry: *Phys. Rev.*, **79**, 393, 476 (1936).
(4) C. S. Wu: *Phys. Rev.*, **77**, 136 (1950).
(5) A. Peres and P. Singer: *Nuovo Cimento*, **15**, 902 (1960).

ics that their conclusions are erroneous, being based on an unpermissible use of classical conceptions concerning the electromagnetic field in the quantum domain.

2. – Summary of essential features of paradox of Einstein, Podolsky and Rosen.

PERES and SINGER accept, as a legitimate example of the paradox of E.P.R., the case of the disintegration of a molecule consisting of two atoms having opposite spin, by some method which does not alter the total spin of the system (6). We shall start therefore by summarizing briefly the main features of this example of the paradox, and we shall show later that there is no essential difference between it and the Wu-Shaknov experiment.

In such a system, the spin state is given, even after the particles are separated, by

$$\psi = 2^{-\frac{1}{2}}[\psi_+(A)\psi_-(B) - \psi_-(A)\psi_+(B)], \tag{1}$$

where $\psi_+(A)\psi_-(B)$ refers to the state in which the particle A has spin $+h/2$ and B has spin $-h/2$, etc. This means that the spins of the two particles are correlated in a manner peculiar to the quantum theory. If the component of the spin of particle A is measured in *any* direction (for example z) then the same component of the spin of particle B is known to be opposite; and since the two particles are far from each other and do not interact, this information has been obtained without in any way disturbing particle B. In classical theory, such a correlation would be easy to understand, because all three components of the spins of each particle are defined simultaneously, and remain opposite to each other, so that the measurements of the spin of A simply gives information about a property already existent and well defined in B. In the quantum theory, however, only one component of the spin of each particles can be defined at a time, and the other two must be ambiguous (subject to uncontrollable « quantum fluctuations »). Therefore, before the spin of particle A in some direction is measured, we cannot suppose that all components of the spin of B are already well defined. On the other hand, it is possible, as we have seen, to choose to measure an arbitrary component of the spin of particle A, and in this way to determine the same component of the spin of B, without any interaction between B and A or between B and the measuring apparatus (after which the other two components of the spin of B will, of course, like those of A, be completely ambiguous). Such a result evidently

(6) See D. BOHM: *Quantum Theory* (New York, 1951), Chap. XXII.

contradicts the notion, commonly accepted before the paradox of E.P.R. was proposed, that the uncertainties of quantum mechanics represent only the effects of disturbances due to the measuring apparatus.

From a discussion of the properties described above, EINSTEIN, PODOLSKY and ROSEN ([2]) came to the conclusion that the quantum mechanics must be an incomplete theory. In doing this, they proposed the following criterion for an element of reality: « If, without in any way disturbing the system, we can predict with certainty the value of a physical quantity, then there exists an element of reality corresponding to this physical quantity ». In the example of the spins, this criterion implies that in a complete theory, there would have to be a set of « elements of reality » corresponding to the simultaneous definition of all three components (and indeed of any component) of the spin of particle B. These elements of reality cannot be described in the quantum theory, because the different components of the spin do not commute. Therefore, some new theory is needed, which would give a more nearly complete description, in the sense that it contained these additional elements of reality.

The above argument was answered by BOHR ([7]), who showed that quantum mechanics implies an inseparability of observing apparatus and observed object which contradicts the criterion of E.P.R. for elements of reality.

3. – A more detailed analysis of the experiment of Wu-Shaknov.

In order to demonstrate that the experiment of Wu-Shaknov is also a valid example of the paradox of E.P.R., we shall first briefly review the discussion of this experiment given in our previous article, and then we shall add a more detailed mathematical analysis, which will help to clarify our answers to the criticisms of PERES and SINGER.

This experiment tests for the correlation of polarization of pairs of photons emitted in the annihilation of positronium. In order to write the wave function for this problem, we first consider a single photon moving in the $+z$ direction. Let ψ_k^x represent the wave function of such a photon, with wave vector k, and linearly polarized in the x direction, ψ_k^y the same for the y direction. Then the wave function of a photon polarized in an arbitrary direction is

$$\psi_k^{\chi} = a_1 \psi_k^x + a_2 \psi_k^y, \tag{2}$$

where $a_1 = \cos\alpha$ and $a_2 = \sin\alpha$. There is a formal analogy here to the spin problem, an analogy that we shall develop in detail presently. Thus, the two

([7]) N. BOHR: *Phys. Rev.*, **98**, 696 (1935).

states ψ_k^x and ψ_k^y correspond to the two opposite spin states φ_+ and φ_- in some direction, say z. The wave function for a spin defined at an arbitrary angle relative to z (in the $x-z$ plane) is

$$\Phi^\beta = \cos\frac{\beta}{2}\Phi_+ + \sin\frac{\beta}{2}\Phi_- \tag{2'}$$

so that Φ^β corresponds to ψ_k^α.

It is clear that a rotation, $\beta = 180°$, of the spin vector corresponds to a rotation, $\alpha = 90°$, of the polarization vector. Therefore, what corresponds to the two possible spin states of an atom are two perpendicular possible directions of polarization of the photons; and just as the component of the spin in the direction β does not, in general, commute with that in the z direction, so the component of the polarization in the direction α does not, in general, commute with that in the x or y directions.

As stated in our article, the wave function of the pair of photons from the annihilation of positronium takes the form

$$\Phi = 2^{-\frac{1}{2}}(\psi_k^x\psi_{k'}^y - \psi_k^y\psi_{k'}^x)\,, \tag{3}$$

where k represents the wave vector of photon A and k' that of photon B (in a direction opposite to that of A).

From the similarity of (1) and (3), one can conclude that there will be a type of correlation in the polarization directions of the two photons, which is analogous to that of the spin directions. We can measure the polarization of photon A in a pair of directions, say x and y. If there is a single photon, then it will be found to be polarized either in the direction x or y. Whatever the direction is, we can deduce that photon B will be polarized in the other direction. This is analogous to measuring one component of the spin of particle A and deducing that the same component of B is opposite. But we can, instead, rotate the pair of axes (x, y) through some angle α and once again we will obtain the same kind of correlation between photons A and B for polarization observables that do not commute with the original set. This is analogous to measuring the spin of particle A in a direction at angle $\beta - 2\alpha$. and obtaining the corresponding correlation to the rotated spin operators for particle B.

At first sight, there may seem to be a difficulty in the above formal analogy between polarization and spin. If one applies the classical idea of a well defined polarization vector too literally, one comes to the conclusion that when the linear polarization in the x and y directions is defined, then the state of polarization cannot be ambiguous. As a result it might seem inconsistent to state that when we rotate the apparatus, new polarization operators not com-

muting with original ones will arise, which represent uncertainties in the polarization vector. It seems likely, indeed, that this problem is at the root of the main criticism of PERES and SINGER (as we shall explain in the next section). In order to clarify this point, we shall therefore develop here a further analysis of how the polarization vector must be described in quantum mechanics.

To show exactly what operators of the electromagnetic fields are measured when a polarization experiment is done, we first write the well known expansion for the vector potential operator, from which the fields can be derived.

$$\boldsymbol{A}(\boldsymbol{x}, t) = \sqrt{2\pi c} \sum_{\boldsymbol{k},i} \left(C_{\boldsymbol{k},i} \exp\left[i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\right] + C^*_{\boldsymbol{k},i} \exp\left[-i(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t)\right]\right) \frac{\varepsilon_{\boldsymbol{k},i}}{k^{\frac{1}{2}}}, \tag{4}$$

where $\varepsilon_{\boldsymbol{k},i}$ is a unit vector normal to $\boldsymbol{k}$ and where i has two values, corresponding to two possible directions of polarization of the k-th wave. $C_{\boldsymbol{k}i}$ and $C^*_{\boldsymbol{k},i}$ satisfy the commutation relation

$$[C^*_{\boldsymbol{k},i},\ C_{\boldsymbol{k}',j}] = \delta_{ij}\,\delta(\boldsymbol{k} - \boldsymbol{k}')\hbar\ . \tag{5}$$

If one introduces the Hermitean operators

$$q_{\boldsymbol{k},i} = \frac{C_{\boldsymbol{k},i} + C^*_{\boldsymbol{k},i}}{\sqrt{2}}, \qquad p_{\boldsymbol{k},i} = \frac{C_{\boldsymbol{k},i} - C_{\boldsymbol{k},i}}{\sqrt{2}}, \tag{6}$$

which satisfy the commutation relations

$$[p_{\boldsymbol{k},i},\ q_{\boldsymbol{k},i}] = -i\hbar\ . \tag{7}$$

We obtain

$$\boldsymbol{A}(\boldsymbol{x}, t) = 2\sqrt{\pi c} \sum_{\boldsymbol{k}} \left[q_{\boldsymbol{k},i} \cos\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t\right) + p_{\boldsymbol{k},i} \sin\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t\right)\right] \frac{\varepsilon_{\boldsymbol{k}\,i}}{k^{\frac{1}{2}}}\ . \tag{8}$$

A linearly polarized photon is represented in the above equation as a superposition over a small range of $\boldsymbol{k}$ with i given a certain value. This leads to a wave packet describing the localization of the electromagnetic energy associated with the photon. If in the Wu-Shaknov experiment, the two photons are far enough apart, we can use wave packets with a broad range of z and therefore with a very small range of k. We can then simplify the problem by approximating the packet by a plane wave having a definite k (as is in fact done in all calculations concerning this experiment). The only part of the vector potential that is relevant for our problem is then a sum of two operators, one for k, the other for k' (which is opposite to k). Let us consider

one of these operators,

$$(9) \qquad \boldsymbol{A}_k(\boldsymbol{x}, t) = \frac{2\sqrt{\pi c}}{k^{\frac{1}{2}}} \left[(q_{\boldsymbol{k}_1} \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) + p_{\boldsymbol{k}_1} \sin(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t))\varepsilon_{\boldsymbol{k}_1} + \right.$$
$$\left. + (q_{\boldsymbol{k}_2} \cos(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t) + p_{\boldsymbol{k}_2} \sin(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t))\varepsilon_{\boldsymbol{k}_2} \right],$$

which represents a photon of direction k polarized in an arbitrary direction.

In classical theory q_{ki} and p_{ki} can be simultaneously well defined, so that the polarization vector for a given k can be specified unambiguously by specifying the four numbers, $q_{\boldsymbol{k}_1}$, $p_{\boldsymbol{k}_1}$, $q_{\boldsymbol{k}_2}$, and $p_{\boldsymbol{k}_2}$. Note that q_{ki} and p_{ki} determine both the intensity of the wave $I_{k,i} = (q_{ki}^2 + p_{ki}^2)/2$ and its phase $\varphi_{\boldsymbol{k}i} = \operatorname{tg}^{-1} q_{\boldsymbol{k}i}/p_{\boldsymbol{k}i}$. With general values of q_k, p_k, one obtains elliptical polarization, which reduces to linear, if only one of the $I_{\boldsymbol{k}}$ is not zero, and to circular, if $I_{\boldsymbol{k}_1} = I_{\boldsymbol{k}_2}$ and $\varphi_{\boldsymbol{k}_1} - \varphi_{\boldsymbol{k}_2} = \pm \pi/2$.

In the quantum theory, q_{ki} and p_{ki} do not commute so that they cannot be defined simultaneously. Thus, as in the case of the spin, the direction of the polarization vector is, in general, ambiguous. In addition, with polarization, we must take into account the phase of the wave, which is also, in general, ambiguous.

Although it would, in principle, be possible to measure q_{ki}, or p_{ki} or some linear function of them, by measuring the Fourier components of the electromagnetic field, this would require that the number of photons became indeterminate. Such an observation is not what is actually done, when polarization is measured optically, nor is it what is done in the Wu-Shaknov experiment. Rather, what is done is to make measurements under conditions in which the number of photons is well defined, so that q_{ki} and p_{ki} must be to some extent ambiguous. What is usually called the « direction of polarization » then correspond only to a kind of average orientation of the field vector itself. In fact, what is measured is I_{ki} which is proportional to the energy of the part of the wave associated with the direction i. Thus, we measure the commuting pair of operators,

$$(10) \qquad \begin{cases} I_{\boldsymbol{k}_1} = \dfrac{p_{\boldsymbol{k}_1}^2 + q_{\boldsymbol{k}_1}^2}{2} \\ I_{\boldsymbol{k}_2} = \dfrac{p_{\boldsymbol{k}_2}^2 + q_{\boldsymbol{k}_2}^2}{2} \end{cases}.$$

Classically, $I_{\boldsymbol{k}_1}$ and $I_{\boldsymbol{k}_2}$ are always well defined numbers with a continuous range of possible values (from 0 to ∞). Quantum-mechanically they need not, in general, be well defined (*i.e.* unambiguous because they are operators), but if the system is an eigenstate of these operators, their values are discrete and restricted to $(n+\frac{1}{2})\hbar$.

In the experiment that we are discussing, we need consider only two possible values of I_{k1} and of I_{k2}, *viz.*, $\hbar/2$ and $3\hbar/2$. The value $\hbar/2$ corresponds

to the ground state (*i.e.*, the vacuum), while $3h/2$ corresponds to the first excited state (*i.e.*, one photon is present). It must be emphasized, however, that the I_{ki} are not components of vectors, rather they are components of tensor of the second rank. Their relation to the field vector is rather indirect. Indeed, when the I_{ki} are defined, the field vector still has a considerable ambiguity, not only in direction, but also in phase, so that it may be thought of as having an indeterminate degree of elliptical polarization. This follows basically from the fact that even the components of $\boldsymbol{A}_k$ with no photon in it still has a «zero point» energy of $h/2$. As a result, the component of A_k is still not zero, but may be said to «fluctuate» in the quantum mechanical sense about an average value of zero. Although this fluctuation is associated with the vacuum properties of the field, it is, nevertheless, of experimental significance, since it would be detected, if a precise measurement of the Fourier component of the field were actually made.

It is clear then that as we have already stated, I_{ki} reflects only some average property of the polarization. In the classical limit, when the number of photons can be very large, the zero point fields can be neglected and the polarization vector approaches a well defined direction. This justifies the usual procedure in the classical domain of identifying the direction of polarization with the orientation of the apparatus by which I_{ki} is measured. In the quantum domain, however, we must be careful not to become confused by the loose application of classical language, which, for the sake of brevity, has customarily been carried over into the description of quantum mechanical experiments.

The fact that after I_{k_1} and I_{k_2} are measured the polarization vector $\boldsymbol{A}_k$ is still ambiguous, is reflected in the experiment in which one measures the operators I'_{k_1} and I'_{k_2} which represent the average polarization along a pair of axes, rotated at an angle α relative to the original set. If the first measurement of I_{k_1} and I_{k_2} were able to determine the polarization vector, $\boldsymbol{A}_k$, unambiguously (as is suggested by the uncritical application of the classical language), then there could be no uncertainty in this vector, so that the result of the measurement in the second set of axes would have been determined without any ambiguities. Actually, however, as we shall now show, the operators, I'_{k_1} and I'_{k_2} do not commute with I_{k_1} and I_{k_2}. They are therefore not determined by I_{k_1} and I_{k_2} without uncertainty. To show this, we recall that $\boldsymbol{A}_k$ transforms as a vector under rotation, so that (q_{k_1}, q_{k_2}) and (p_{k_1}, p_{k_2}) each separately behave as a vector. We therefore have

$$
\begin{cases}
q_{k_1} = q'_{k_1}\cos\alpha + q'_{k_2}\sin\alpha \\
q_{k_2} = q'_{k_2}\cos\alpha - q'_{k_1}\sin\alpha \\
p_{k_1} = p'_{k_1}\cos\alpha + p'_{k_2}\sin\alpha \\
p_{k_2} = p'_{k_1}\cos\alpha - p'_{k_1}\sin\alpha
\end{cases}
\tag{10}
$$

and from this, we obtain

$$(11)\qquad \begin{cases} I'_{k_1} = \dfrac{(p'^2_{k_1} + q'^2_{k_1})}{2} = I_{k_1}\cos^2\alpha + I_{k_2}\sin^2\alpha + \sin 2\alpha(p_{k_1}q_{k_1} + p_{k_2}q_{k_2})\,, \\ I'_{k_2} = \dfrac{p'^2_{k_2} + q'^2_{k_2}}{2} = I_{k_1}\sin^2\alpha + I_{k_2}\cos^2\alpha - \sin 2\alpha(p_{k_1}q_{k_1} + p_{k_2}q_{k_2})\,. \end{cases}$$

(Note that the I'_{ki} do not transform as a vector.)

It is clear that I'_{k_1} and I'_{k_2}, which both contain the operators $(p_{k_1}q_{k_1} + p_{k_2}q_{k_2})$, do not commute with I_{k_1} and I_{k_2}. This demonstrates that the measurement of what is usually called « polarization » in a pair of orthogonal directions does not completely determine the result of a similar measurement in another pair of orthogonal directions.

In our original article, we have indicated this property of the polarization (*) but without going into as much detail as we did here. As we pointed out there one could already infer this property from equation (2) of the present article which states that the wave function (and not the I_{ki} operator) transforms as a vector. Whenever any measurement whatsoever yields two possible results, corresponding to orthogonal wave functions, then a linear combination of these wave functions represents an eigenstate of another operator, (or commuting set of operators), which fails to commute with the first operator, (or commuting set of operators). Since the polarization measurement in a rotated frame leads to just such a linear combination of wave functions, we concluded that the measurement of what is usually called linear polarization in the rotated frame corresponds to a set of operators that do not commute with the original set. A more familiar case of this behaviour is found with spin, where as equation (1) shows, the measurement of a given component of spin in a rotated frame leads to a similar linear combination of eigenfunctions corresponding to measurements in the original frame.

On the basis of the above discussion, we see that the experiment of Wu-Shaknov is an example of the paradox of E.P.R. As in the case of spins, we can here measure what is usually called the « linear polarization » of one of the photons, say A, and from this, we conclude that B has an orthogonal « linear polarization ». This can be done with the orthogonal pair of directions (x, y), equivalent to finding that the spin is $\pm h/2$ in the z direction; or else is can be done in any rotated pair of orthogonal directions (equivalent to measuring the component of the spin along a rotated axis). Since the operators corresponding to these sets of measurements do not commute, we have again

(*) See reference [1], p. 1073, first and second paragraph after equation (4).

the same kind of quantum mechanical correlation which was described in Section 2 in connection with the spin, and which is at the basis of the paradox of E.P.R.

4. – Discussion of criticism of Perez and Singer.

As we stated in the introduction to this article, the criticisms of the conclusions of our previous paper by PERES and SINGER are based on an erroneous application of classical conceptions of polarization of electromagnetic waves in the quantum domain. This is brought out most clearly in their main criticism (*), which occurs near the beginning of Section 3 of their article ([5]). They state that « linear polarization is useless (for the purpose of representing the paradox E.P.R. in terms of photons) because the two *directions* of linear polarization are similar to the two *values* of the spin in some given direction. Thus, obviously there can exist no uncertainty relations for linear polarization alone. »

The above quotation clearly implies that PERES and SINGER regard the usual linear polarization experiments as capable of determining the direction of linear polarization unambiguously.

As we have shown in Section 3, however, such an implication is erroneous, because these experiments do not measure the field vector, but only certain functions I_{k_1} and I_{k_2}, which still leave a great deal of ambiguity in the phase, magnitude, and direction of this vector, as well as in the degree of ellipticity of the polarization. This ambiguity is reflected, as we have shown, in the fact that measurements of « linear polarization » in some other pair of directions does not commute with that in the original pair.

Because of the above misconception concerning the nature of polarization in the quantum domain, PERES and SINGER went on to suggest an alternative formulation of the paradox E.P.R. for photons, which they (also erroneously) claimed to be inadmissible. To do this, they introduce what are called « Stokes operators » which satisfy commutation rules (**) similar to that of spin. One of these operators corresponds to the circular polarization of a photon, and

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(*) PERES and SINGER begin this section with a statement that we « have overlooked the important fact that the polarization of photons is physically different from the spin of fermions, as photons have zero at mass, their spin... is always oriented in the direction of propagation. » Since we have never attempted to formulate the paradox of E.P.R. by considering directly the spin operator of photons, the above statement has no connection with our article, and is indeed quite irrevelant to the point at issue.

(**) See reference ([5]), equation (7).

another to its linear polarization. Because these operators do not commute, one can obtain the result already known by other methods that linear and circular polarization cannot be defined together. They then assert that this set of operators cannot be used to provide an example of the paradox of E.P.R. For precisely defined elements of reality would then have to exist in photon B, corresponding to the simultaneous definition of the states of linear and circular polarization. This, they say is « non-sensical because if the circular polarization of a photon is precisely defined, its linear polarization cannot be precisely defined, and vice-versa. » Thus, they conclude that the attempt to regard the Wu-Shaknov experiment as an example of the paradox of E.P.R. « does not lead to a paradox, but to an inconsistency ».

The above argument is likewise based on the unjustifiable use of the classical description of polarization in the quantum domain. Indeed, from their statement that there can exist no uncertainty relations for linear polarization alone, it would follow that once the wave is defined as « linearly polarized », its field vector would have a well defined direction, so that by definition it could not at the same time be circularly polarized. The error in this point of view is, as we have already pointed out, that the measurement of « linear polarization » does not define the field vector without uncertainties, but leaves a residual fluctuating part of undefined phase and amplitude. In other words, a better idea of this field vector is obtained by regarding it as elliptically polarized, with an indeterminate degree of ellipticity, and with an *average* polarization in the direction of the measurement. Clearly, such an elliptically polarized wave can be regarded as made up of a linearly polarized wave plus a circularly polarized wave. In the classical limit, both could be well defined and measured together, so that it would have meaning to specify the intensities and phases of both, without any self-contradiction. Quantum-mechanically, however, the two kinds of polarization cannot, as we shall see, be measured together, so that if one is well defined the other must be ambiguous.

We shall now discuss this problem more formally. In order to define the field vector completely, we need (as shown in Section 3) four operators, of which only two can be defined together. Instead of the original set (p_{k_i}, q_{k_i}), we can take the intensities, I_{k_1} and I_{k_2}, corresponding to « linear polarization » and another set, I_{k_+} and I_{k_-}, corresponding to « circular polarization ». These latter are defined by

$$\text{(10)} \qquad \begin{cases} I_{k_+} = \left[\dfrac{(q_{k_1} - p_{k_2})^2 + (q_{k_2} + p_{k_1})^2}{2}\right] = \dfrac{I_{k_1} + I_{k_2} + (p_{k_1} q_{k_2} - p_{k_2} q_{k_1})}{2} \\ I_{k_-} = \left[\dfrac{(q_{k_1} + p_{k_2})^2 + (p_{k_1} - q_{k_2})^2}{2}\right] = \dfrac{I_{k_1} + I_{k_2} + (p_{k_1} q_{k_2} - p_{k_2} q_{k_1})}{2} \end{cases}$$

It is clear that I_{k_1} and I_{k_2} do not commute with I_{k_+} and I_{k_-}.

On photon A, we can then measure either the set I_{k_1} and I_{k_2} or the set I_{k_+} and I_{k_-} (which classically could, of course, be defined together). Thus, as in the case of spin, we have a set of non-commuting operators, subject to correlations at long distances, without any interactions. The analogy is that when « linear polarization » is measured, the « circular polarization » is indeterminate and vice-versa; while with spin, when one component is measured, the others are indeterminate and vice-versa. The fact that there are three components of the spin and only two for polarization is evidently not relevant here, nor is it relevant that polarization has phase as well as amplitude, while spin has no phase.

After discussing the above described example, PERES and SINGER state further that « it seems that the impossibility of constructing an E.P.R. paradox for photons is connected with the impossibility of describing them without second quantization, while *this paradox can be raised only for that part of quantum mechanics having a classical counterpart.* »

With reference to the above statement, we would like to emphasize that the essential characteristic of the paradox of E.P.R. is that there must be two or more non-interacting dynamical systems separated in space so that it will be certain that if one of these variables is measured, the other is not disturbed in any way. Then, if the properties of the above systems are correlated in the way that has been described (such that the measurement of one of a set of non-commuting observables of one of them provides the value of the corresponding observable in the other), then we have an example of the paradox. In the example of the Wu-Shaknov experiment, we describe the systems in terms of a pair of wave packets of the vector potential operator, representing photons, which are clearly very distant from each other in space, and which are not connected with each other in any way, while the measurements under discussion are taking place.

Finally, it should be stated that the quantum theory of the electromagnetic field, (*i.e.*, the theory involving second quantization) does evidently have a classical counterpart, namely, the classical theory of this field. Indeed, in defining our polarization operators we were guided by the classical limit, replacing dynamical variables of the field vectors by operators, etc. The only difference between our example and that given in the original paper of E.P.R. is that we have used a field example, while there, a particle example was used. But as shown above, this difference is not relevant.

Elsewhere in their article (*), PERES and SINGER criticize our conclusions along a different line, claiming that our analysis of the Wu-Shaknov experiment would, in any case, not be of importance, because the conclusions that

(*) See reference ([5]), section (2).

we drew could have been obtained immediately by considering the problem of parity conservation in electrodynamics.

Briefly, this point is concerned with the Furry hypothesis (*), which was aimed at avoiding the « paradoxical » features of the quantum mechanical treatment of this problem. This hypothesis involved the assumption that the many-body Schrödinger equation is correct for atomic orders of distance (where it has been tested quite well), but breaks down in a fundamental way at macroscopic distances (where the paradox of E.P.R. is relevant and where there has previously been no clear experimental test). Briefly, FURRY considers the possibility that in the latter case, the wave function for the system becomes a product of the wave functions of the parts, with no superposition, but with a certain probability that a particular product will appear. In our original article we calculated the results of the Wu-Shaknov experiment according to the Furry hypothesis with all possible assumptions concerning the polarization states with which the system separates, and we have shown that they are all inconsistent with the results of the experiment (which were already known to agree within experimental error with the predictions of the quantum theory).

PERES and SINGER remark that for the case of a pair of photons, the Furry hypothesis would lead to non-conservation of parity. They then assert that if parity were not conserved, this would have been noticed long ago, because quantum electrodynamics allows experiments of the highest accuracy.

It must be stated, however, that no test of conservation of parity in electrodynamics with regard to macroscopic orders of distances was possible, previous to the Wu-Shaknov experiment (and similar experiments). First of all, it is evident that no purely classical observations of the electromagnetic field can possibly test for the conservation of parity. For the parity is defined as a reflection property of the *wave function* of the whole system (in this case the electromagnetic field). Since the wave function does not appear in the classical limit, no classical property can depend on the parity of this function. The fact (to which PERES and SINGER have alluded) that parity is a discrete quantum number, for which there is no meaning to statistical conservation, is therefore not relevant in the classical limit.

Thus far, parity conservation has been tested most accurately in spectroscopy, but here the many-body systems (*e.g.* atoms) have not extended over very long distances, so that these experiments do not test for the possibility of a breakdown of conservation of parity in the case of the paradox of E.P.R. The only way to test for this breakdown is to consider an experiment carried out to a quantum-mechanical level of accuracy on a many-body problem, in

(*) See reference ([3]).

which there is a correlation of the properties of systems at a macroscopic order of distance; and as we have already stated, experiments of the general type that we have described here are the first case where such a test can be made.

RIASSUNTO (*)

In un lavoro precedente abbiamo suggerito come prova per il paradosso di Einstein, Podolsky e Rosen, l'esperimento di WU-SHAKNOV sulla radiazione di annichilazione del positronio. In questo articolo rispondiamo ad alcune critiche alle nostre conclusioni, avanzate da PERES e SINGER. Si mostra come queste critiche siano erronee, essendo basate su una scorretta interpretazione della polarizzazione della radiazione elettromagnetica nel dominio quantistico.

(*) *Traduzione a cura della Redazione.*

LETTERE ALLA REDAZIONE

(La responsabilità scientifica degli scritti inseriti in questa rubrica è completamente lasciata dalla Direzione del periodico ai singoli autori).

Increase in the Intensity of Cosmic Radiation on May 4-th, 1960.

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(ricevuto il 24 Maggio 1960)

During the recent solar injection on May 4-th, 1960, a standard IGY type neutron monitor was in operation in these laboratories. The pressure corrected intensity record obtained between 0300 and 2200 h UT is shown in Fig. 1, where the data have been grouped into hourly ($\sigma = 0.5\%$) and 15 minute intervals. Part of this record, on a more open time scale is given in Fig. 2 covering the period 1001 to 1216 h UT in four minute intervals and shows the percentage increase above the normal intensity. Superimposed on our data in Fig. 2 are the neutron intensity increases (scaled down by a factor of 33) reported by Steljes and Carmichael [1] from Deep River.

The increase at Leeds was small, with a primary peak $\sim 8\%$ above normal, which makes the precise determination of the onset time difficult. However in Fig. 2, the intensity in the interval 1033-1037 h appears to have risen by more than two standard deviations above normal and this interval overlaps the first period of increase, 1030-1035 h, in the record from Deep River. Our data therefore show no evidence of any delay between the times of arrival of the injected particles for these two stations. Also from Fig. 2 it appears that, within the statistical uncertainty of the data, the peak intensity occurred at the same time for both stations and up to 1055 h the intensities decreased in a similar manner. These observations, together with the evidence for similar onset times, suggest that for this initial period « prompt » particles were reaching Leeds although the peak increase here was lower by a factor of about 30 than at Deep River. There is some indication that the decrease is more rapid at Leeds than at Deep River, although it is doubtful whether this is significant. Its interpretation would be that Leeds lay in part of an impact zone for which the minimum rigidity for entry was higher than for the zone affecting the Canadian station. The low intensity suggests that Leeds was very much on the edge of the zone.

The correspondence of the data plotted in Fig. 2, ceases at about 1055 h, thereafter the Leeds data show an increase and continue to fluctuate at $\sim 7\%$ above normal until at least 1200 h, whilst the Deep River intensity continues to fall smoothly. By analogy with the 1956 event, this period would be

[1] J. F. Steljes and H. Carmichael: private communication (1960).

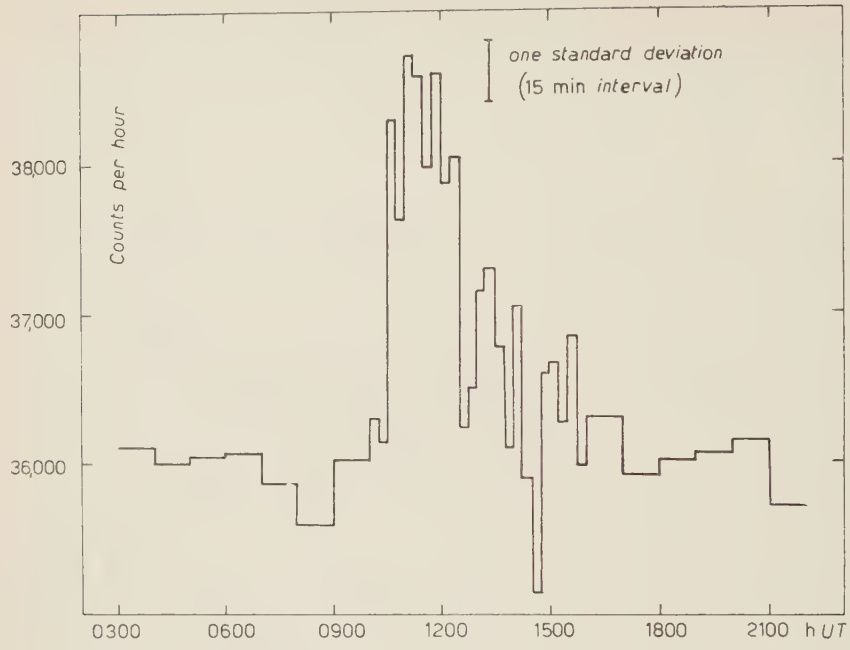

Fig. 1. – Intensity of the nucleonic component, May 4-th 1960, Leeds.

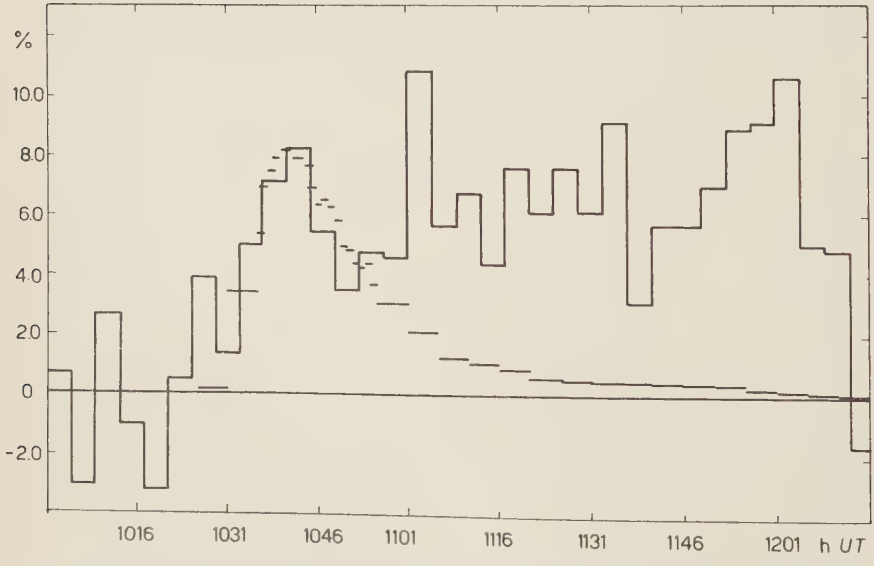

Fig. 2. – Variation of the nucleon intensity, May 4-th 1960. Leeds data shown by histogram. Deep River data (scaled down by a factor of 33) shown by dashed curve.

expected to relate to the « diffuse phase » in which the particles would arrive isotropically, though on this occasion, this phase may represent a much more complex situation. However, it does appear certain that the intensity during the diffuse phase is relatively much weaker as compared with the peak intensity of the initial phase than was the case in 1956. If the Leeds observation between 1100 and 1200 h (Fig. 2) represents the maximum intensity of this « diffuse phase » it is only about one twenty-fifth of the Deep River peak as compared with a ratio of about one half in 1956 between comparable stations.

However, the degree of isotropy exhibited during this phase is not so strongly evident as in 1956. On that occasion widely separated stations showed a striking uniformity of response lasting more than 12 hours (MARSDEN and WILSON [2]). For the present event, between 1100 and 1200 h, the Deep River intensity decreased smoothly from 5% to 5% above normal, whilst the Leeds intensity fluctuated about a level $\sim 7\%$ higher than normal. Further, between 1230 and 1245 h, (Fig. 1), there is a large statistically significant discontinuity.

In view of the differences reported here during the second phase it may be noted that in 1956, the flare occurred close to the minimum of a Forbush decrease whereas for the present event, although there was a fairly large decrease on 30-th April, the intensity had recovered almost to its pre-storm value before the onset of the disturbance.

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[2] P. L. MARSDEN and J. G. WILSON: *Suppl. Nuovo Cimento*, 8, 228 (1958).

An Example of the Reaction: $K^- + {}^{12}C \to {}^{9}Be_\Lambda + \pi^- + {}^{3}He$ and the Subsequent Decay in Flight of the ^{9}Be Hyperfragment (*).

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(ricevuto il 29 Giugno 1960)

During a systematic scan of an emulsion stack exposed in a high intensity pulsed magnetic field to the separated K^--meson beam of the Berkeley Bevatron, an interesting event was found in which all the products from a K^- capture star are visible. A projection drawing of the event is shown in Fig. 1. Two of the three particles produced in star A have been identified as follows: Track (a) by b^* *vs* range

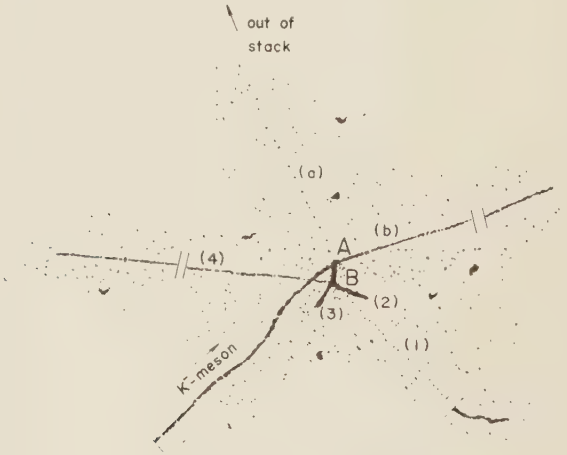

Fig. 1. – Drawing of the event projected onto the emulsion plane.

$$K^- + {}^{12}C \to \pi^- + {}^{3}He + {}^{9}Be_\Lambda$$
$${}^{9}Be_\Lambda \to \pi^- + p + {}^{8}Be$$
$${}^{8}Be \to 2\alpha\,.$$

K^- is captured by a ^{12}C-nucleus at point A. $^{9}Be_\Lambda$ (track AB) decays in flight at point B. Tracks (a) and (1) represent the pions, (b) the ^{3}He-particle, (4) the proton; and (2) and (3) the two α-particles.

(*) Partially supported by the U.S. Atomic Energy Commission.

measurements as a π-meson. Unfortunately, it is not arrested in the stack; a blob count indicates a pion energy of (120 ± 20) MeV at star A ([1]). Track (*b*) has a total range of 113.8 μm and is identified by momentum balance as ^{3}He.

Momentum balances in all directions if the hyperfragment has a momentum of (410 ± 25) MeV/c at emission. The total energy released in star *A* is (490 ± 20) MeV if track (*b*) is ^{3}He; close to the rest mass of the K^--meson. It therefore appears that no neutral particles were involved as would be initially suspected from the momentum balance obtained.

Star B contained four particles. Track 1 is a π^--meson which is arrested in the stack having a total residual range at emission of (3.95 ± 0.04) mm. Track 4 is a proton of 8.82 MeV. Tracks (2) and (3) have a probable charge 2 and could be either ^{3}He or ^{4}He. Heavier nuclei of greater charge would give negative binding energies for the Λ^0-hyperon in the fragment. Momentum balances in two directions at right angles to the hyperfragment for any of the above combinations.

Four assumptions are possible for tracks 2 and 3:

1) Both ^{3}He: This would make the hyperfragment $^7Be_\Lambda$: The binding energy of the Λ^0-hyperon would be -6.4 MeV.

2) Both ^{4}He. The hyperfragment would be $^9Be_\Lambda$ and the binding energy of the Λ^0-hyperon in the fragment, (6.2 ± 0.4) MeV.

3) Track 2 is ^{4}He and track 3 is ^{3}He: the hyperfragment is $^8Be_\Lambda$ and the binding energy of the Λ^0-hyperon only (0.6 ± 0.4) MeV.

4) Track 3 is ^{4}He and track 2 is ^{3}He: the hyperfragment is again ^{8}Be and the binding energy (4.5 ± 0.5) MeV.

We can rule out (1), (3) and possibility of emission of neutral particles on the grounds of too low a binding energy for the Λ^0-hyperon. This leaves us with possibilities (2) and (4). We prefer (2) for the following reasons: first of all, the binding energy of the Λ^0-particle in case (4) disagrees with other known examples ([3]) of $^8Be_\Lambda$. Secondly, it is not unreasonable to assume that the decay was of the form:

$$^9Be_\Lambda \to \pi^- + p + {}^8Be; \quad {}^8Be \to 2\alpha .$$

If this were the case, the ^{8}Be nucleus would either decay from its ground or first excited state. The energy of tracks 2 and 3 in the c.m. system of a supposed ^{8}Be nucleus is calculated to be 0.14 MeV, indicating a possible decay from the ground state (98 keV above 2 alpha particles). In addition the ^{8}Be direction would be essentially the same as the ^{9}Be direction (it having zero velocity in the c.m. system of the ^{9}Be); this might be expected from this type of decay if the impulse approximation is valid.

Momentum balances in the hyperfragment direction if the fragment decays in flight with a total momentum (292 ± 17) MeV/c at decay. This would correspond to momentum at emission of (394 ± 19) MeV/c for $^9Be_\Lambda$ which is within the experimental error of the value needed to obtain a momentum balance in the primary star. This additional fact emphasizes the validity of our assumption.

The potential time of flight of the hyperfragment was $5.9 \cdot 10^{-13}$ and the actual time of flight $1.3 \cdot 10^{-13}$ s. There has been one previous report of a $^9Be_\Lambda$ decay in flight ([2]) for which the potential

([1]) G. ALEXANDER and R. H. W. JOHNSTON: *Nuovo Cimento*, **5**, 363 (1957).

([2]) W. E. SLATER and PH. D. THESIS: *Suppl. Nuovo Cimento*, **10**, 1 (1958).

and actual flight times were $2.3 \cdot 10^{-12}$ and $1.5 \cdot 10^{-12}$ s respectively.

To summarize: we believe that the best explanation of the event is that it is an example of $K^- + {}^{12}C \to {}^{9}Be_\Lambda + {} + \pi^- + {}^{3}He$ followed by the decay in flight of the hyperfragment ${}^{9}Be_\Lambda \to p + \pi^- + {}^{8}Be$; ${}^{8}Be \to 2\alpha$. The binding energy of the Λ^0-hyperon for this case is $(6.22 \pm .44)$ MeV and momentum can be balanced in all directions if the decay occurs in flight. This binding energy is in good agreement with the six other known examples of ${}^{9}Be$ decay which have been summarized by LEVI-SETTI, SLATER and TELEGDI [3] and yield an average value of $(6.6 + .31)$ MeV. It is likely that the ${}^{9}Be_\Lambda$ decayed into ${}^{8}Be$ initially because the α-particle energies are such that the decay could have occured from the ground state of ${}^{8}Be$ and the velocity of the ${}^{8}Be$ in the c.m. system of the ${}^{9}Be$ is zero.

* * *

The author would like to thank the members of the UCLA high energy group for encouragement and advice, in particular Professors W. E. SLATER, D. H. STORK, D. J. PROWSE and H. K. TICHO. Miss J. NAFTULIN and Mrs. C. SWAN are gratefully thanked for some of the microscope work, Dr. E. J LOFGREN for making the exposure possible and Dr. R. S. WHITE and his group at Livemore for generous exposure facilities.

[3] R. LEVI-SETTI, W. E. SLATER and V. L. TELEGDI: *Suppl. Nuovo Cimento*, **10**, 68 (1958).

IL NUOVO CIMENTO Vol. XVII, N. 6 16 Settembre 1960

Relativistic Increase in Bubble Density in a $CBrF_3$ Bubble Chamber.

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(ricevuto il 4 Luglio 1960)

Bubble density measurements along tracks in a sensitivity stabilized bubble chamber [1] can yield reliable information on the particle velocity β. For $0.2<\beta<0.8$, the bubble density is known [2,3] to be roughly proportional to $1/\beta^2$. For highly relativistic electrons some evidence has been found for a « relativistic increase » of the bubble density in propane and carbon dioxide-propane by Blinov *et al.* [2], and by Argan *et al.* [4] respectively. We present here some results on bubble density measurements of high energy pion and proton tracks as observed in a two liter sensitivity stabilized bubble chamber filled with $CBrF_3$.

The chamber was exposed to a 16 GeV/c π^--beam of the CERN proton synchrotron. The bubble density of the primary pion tracks ($\gamma=115, \gamma=(1-\beta^2)^{-\frac{1}{2}}$), has been compared to the bubble density of V-particle tracks. The particles produced in the two-body decay modes of the K^0-mesons ($K^0\to\pi^++\pi^-$), and of the Λ^0 hyperons ($\Lambda^0\to p+\pi^-$) were identified and their bubble density and their energy was determined. The bubble densities have been measured both by the gap length distribution method as well as by simple bubble counting corrected for counting losses. In order to identify the particles, and to calculate their energies, the true space coordinates were reconstructed by a method described by Borelli *et al.* [5]. It was first verified that the V-particle was coplanar with the origin of the strange particle production. From angle considerations alone the particles in a V can be interpreted as (π,π), (p,π) or (π,p). In all events considered here one of the two latter cases always could be excluded by range considerations of the proton. The two remaining possibilities could be distinguished in approximately 90% of all cases by multiple scattering measurements.

(1) B. Hahn, A. W. Knudsen and E. Hugentobler: *Suppl. Nuovo Cimento*, **15**, 236 (1960).

(2) G. A. Blinov, Iu. S. Krestnikov and M. F. Lomanov: *Žurn. Èksp. Teor. Fiz.*, **31**, 762 (1956); (translation: *Sov. Phys. Journ. Exp. Theor. Phys.*, **4**, 661 (1957)).

(3) W. J. Willis, E. C. Fowler and D. C. Rahm: *Phys. Rev.*, **108**, 1046 (1957).

(4) P. E. Argan, A. Gigli, E. Picasso and G. Tomasini: *Nuovo Cimento*, **10**, 177 (1958).

(5) V. Borelli, P. Franzini, I. Mannelli, A. Minguzzi-Ranzi, R. Santangelo, F. Saporetti, V. Silvestrini, P. Waloschek and V. Zoboli: *Nuovo Cimento*, **10**, 525 (1958).

The results obtained from the evaluation of fourteen K^0 decays, and four Λ^0 decays are shown in Fig. 1. The relativistic increase of bubble density in $CBrF_3$ has been obtained by bubble density and multiple scattering meas-

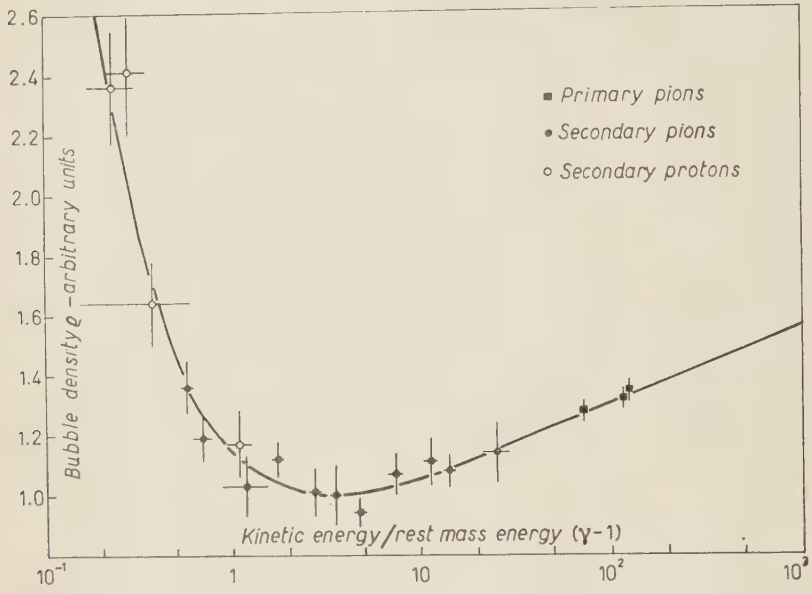

Fig. 1. – Bubble density versus $\gamma - 1$ for high energy pions and protons. ■, ●, ○ refer to primary pions, secondary pions and secondary protons. Explanation of solid line see text.

bubble density ϱ is plotted versus the quantity $\gamma - 1$. The errors indicated for the bubble densities are statistical errors. The errors for the quantities $\gamma - 1$, have been computed under the assumption that the accuracy in angular determinations by the mentioned projection method is $\pm 1°$. Points from pions with overlapping errors in energy have been averaged in the graph. The points at $\gamma - 1 = 71$, 114, and 121 correspond to primary pions of known momenta of 10, 16, and 17 GeV/c. The bubble density in the minimum has been arbitrarily normalized to unity. The results show a relativistic increase in bubble density of approximately $(30 \pm 5)\%$ from minimum bubble density to the bubble density at $\gamma = 100$.

Additional information confirming the urements on star shower particles and on high energy electrons.

The experimental results have been tentatively compared to a theoretical curve of the form

$$\varrho = \frac{0.056}{\beta^2} [\log (\gamma^2 - 1) + 15 - \beta^2] ,$$

as shown in Fig. 1. It may be noted that the number of δ-rays of energy comparable to the mean ionization potential of the atoms produced by a primary particle of energy γ is also of this form ([6]). Still little is known on the mechanism of bubble formation, and no conclusion can be drawn from

([6]) H. BETHE: *Hand. d. Phys.*, **24**, 516 (1933).

this result alone. However, if only δ-rays with energy much larger than the mean ionization potential were responsible for bubble formation, no relativistic increase would be expected.

It is probable that the relativistic increase is dependent on the chamber liquid and the operating conditions. Only $CBrF_3$ and a particular set of conditions have been investigated.

* * *

The authors wish to thank the members of the machine group and to the Hydrogen Bubble Chamber group of CERN for kind collaboration.

Investigation of an Electron-Photon Cascade of more than 10^{13} eV (*).

D. M. Haskin, E. Lohrmann, M. W. Teucher (**) and M. Schein (***)

Department of Physics, University of Chicago - Chicago, Ill.

(ricevuto il 4 Luglio 1960)

In this letter the development of an electron-photon cascade of $2.3 \cdot 10^{13}$ eV in nuclear emulsion very near the shower axis and before the cascade maximum will be described. Good agreement with the shower theory of Nishimura and Kamata [1] is obtained.

In a stack of nuclear emulsion, 60 cm × 30 cm, 200 sheets, which was exposed in Texas above 11 0000 feet, a very dense cascade shower was found. It was

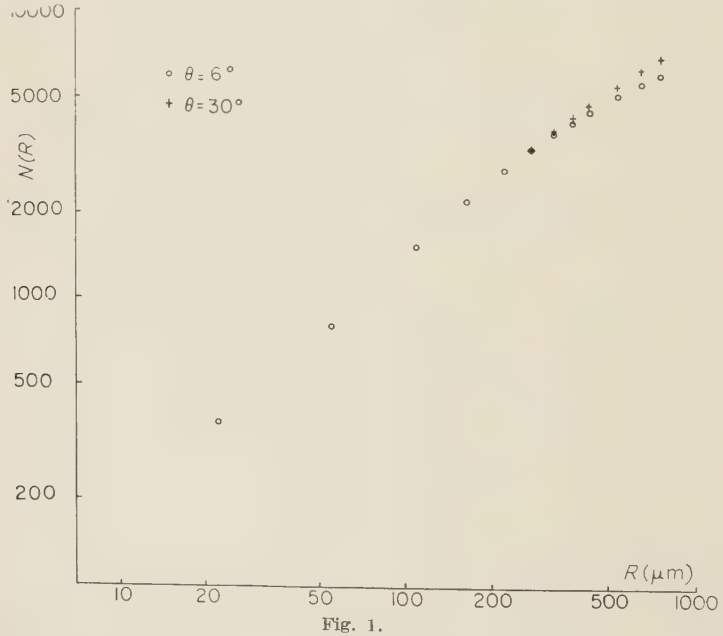

Fig. 1.

(*) Supported in part by the National Science Foundation, and the joint program of the Office of Naval Research and the Atomic Energy Commission.

(**) Present address: Physikalisches Staatsinstitut, University of Hamburg.

(***) This manuscript was completed after Prof. Schein's death. Any error, therefore, is due to the other authors only.

(1) J. Nishimura and K. Kamata: *Progr. Theor. Phys.*, **5**, 889 (1950); **7**, 185 (1952); K. Kamata and J. Nishimura: *Prog. Theor. Phys. Suppl.*, **6**, 93 (1958).

traced back to a nuclear interaction of type 6 + 16p. The cascade originating from this event is available for analysis in the stack for 22 cm. Since no double or multiple core structure of the cascade was found (within a resolution of about $2 \cdot 10^{-5}$ rad.), it is concluded, that it is either originated by a single γ-ray or by two γ-rays from the same π^0-meson. This conclusion is supported by the fact, that only three charged mesons are emitted from the primary interaction under small angles ($< 10^{-3}$ rad.). For the same reason, the effects of secondary interactions of the shower particles on the cascade development can be neglected. The shower can therefore be treated as a pure electron-photon cascade. This was explained in more detail in a previous paper [2]. There the energy of the electron shower was estimated to be $2.5 \cdot 10^{13}$ eV from a comparison with shower theory.

In the meantime more extended and more accurate numerical calculations of the lateral and longitudinal cascade development were carried out by NISHIMURA [3], which allow a much more detailed and meaningful analysis of our experimental data.

Target diagrams of the shower were made at distances t of 4.0, 5.6 and 7.2 cascade units (c.u.) from the origin of the electronic shower. One c.u. in emulsion equals 2.9 cm. The origin was uncertain within about $\pm .1$ c.u. All tracks having a projected angle with the shower axis $< \theta$ were included Two values of θ were used, namely 6° and 30°. A correction for the background of random tracks was applied. Fig. 1 gives the distribution of the number $N(R)$ of tracks lying inside a radius R around the shower axis at $t=7.2$ c.u. Fig. 1 allows a comparison of the results obtained for the two different choices of θ. All target diagrams showed cylindrical symmetry around the shower axis within the statistical accuracy.

The comparison between the theoretical distribution of $N(R)$ according to NISHIMURA and our measurements is shown in Fig. 2 at $t=4.0$ c.u., 5.6 c.u. and 7.2 c.u. from the shower origin.

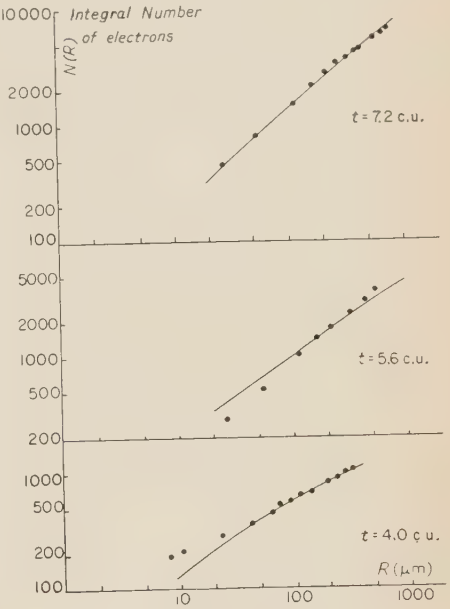

Fig. 2.

It was assumed that our shower was initiated by a single γ-ray. The results would be changed very little if the shower would be started by two γ-rays of the same total energy. Since in the theory electrons down to the smallest energies are included, the measurements employing the $\theta=30°$ criterion were used for the comparison.

([2]) M. W. TEUCHER, E. LOHRMANN, D. M. HASKIN and M. SCHEIN: *Phys. Rev. Lett.*, **2**, 313 (1959).

([3]) J. NISHIMURA and J. KIDD: for use by the *International Cooperative Emulsion Flight Project*, prepared at the request of the late Professor M. SCHEIN.

([4]) See, for example, K. PINKAU: *Nuovo Cimento*, **3**, 1285 (1956). There, the case is discussed for two electrons of equal energy starting the cascade.

The primary energy E_0 of the shower is the only adjustable parameter. The best fit is obtained for $E_0 = 2.3 \cdot 10^{13}$ eV. A different choice of E_0 would result in a parallel displacement of the theoretical curve along the R-axis, because for $R \ll 1$ c.u. $N(R)$ depends on $E_0 R$ only.

A good agreement between theory and experiment is obtained for all three distributions at values of $t = 4.0$, 5.6 and 7.2 c.u. The small disagreement for the smallest value of R can be attributed to the experimental difficulty of resolving the tracks in the core of the shower. At $t = 4.0$ c.u. and $t = 5.6$ c.u. the number of tracks in the innermost core could only be determined by measuring the total ionization. The uncertainty introduced by this is large enough to account for the observed deviation from theory.

These observations, being carried out very near the shower axis and before the cascade maximum, show that the distribution of the high energy electrons in a cascade is in agreement with the predictions of cascade theory ([1,2]).

* * *

The authors would like to thank Prof. J. NISHIMURA for many valuable discussions and for undertaking the tedious task of numerical evaluation of his cascade theory and J. KIDD, who assisted with the calculations.

Alpha and Beta Bands in KCl and NaCl (*).

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(ricevuto il 4 Agosto 1960)

The α and β bands have been studied experimentally in KI, KBr and NaBr since 1951 ([1-3]) and are attributed to exciton production in the vicinity of an anion vacancy and an F-center respectively. Ruchardt's extensive studies of the α band in KBr have led to a qualitative model for the X-ray production of anion vacancies and for their thermal destruction. His experimental results on NaBr and KI are difficult to describe with this model which thus may not be appropriate for all alkali halides. An extension of Ruchardt's studies to KCl and NaCl, whose other properties are better understood, has been undertaken in order to study anion vacancy production and destruction in these crystals. The α an β bands have not yet been observed in KCl and NaCl because they lie in a wavelength region not accessible to common ultraviolet absorption spectrometers.

In our work, the α and β bands were observed in thin cleaved samples of Harshaw KCl and NaCl single crystals using a Hilger 1 m vacuum grating monochromator with a hydrogen capillary discharge ([4]) as a source and a photomultiplier detector.

Figs. 1 and 2 show typical absorption spectra of KCl and NaCl crystals X-rayed at 95 °K and Table I gives the position of the peaks of the two bands. Additive coloration produced only a β band which could be reduced by irradiation with F-light at 95 °K, the α band growing during this process. The β band could also be produced readily by the undispersed ultraviolet irradiation from the

Table I. – *Locations of α and β bands at 95 °K (Å).*

	α	β
KCl	1 770	1 664
NaCl	1 743	1 682

(*) Partially supported by the Air Force Research Division of the A.R.D.C., U.S. Air Force, through its European Office.

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([1]) C. J. Delbecq, P. Pringsheim and P. Yuster: *Journ. Chem. Phys.*, **19**, 574 (1951).

([2]) W. Martienssen: *Zeits. Phys.*, **131**, 488, (1952).

([3]) H. Ruchardt: *Zeits. Phys.*, **140**, 574 (1954); *Phys. Rev.*, **103**, 873 (1956).

([4]) P. L. Hartman and J. R. Nelson: *Journ. Opt. Soc.*, **47**, 646 (1957).

1 kW hydrogen discharge used as a source. The thermal stability of anion vacancies produced in a KCl crystal by

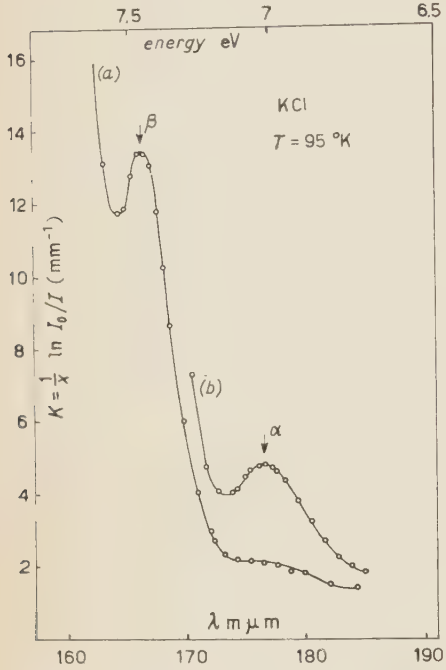

Fig. 1. – Curve *a*): The absorption spectrum of a crystal of KCl X-rayed at 95 °K for 30 minutes (50 kV, 20 mA). Curve *b*): The α band in a more heavily darkened crystal.

irradiation at 95 °K is shown in Fig. 3, where the ordinate gives the fraction of vacancies still present at a given temperature.

There have been few theoretical studies of these bands. Using the charge transfer model, BASSANI and INCHAUSPÉ [5] calculated the displacements of the α and β bands from the first fundamental absorption peak. Their results are given in Table II along with experimental values taken from the sources indicated. For the chlorides $h\nu_{free}$ was taken as the peak of the longer wavelength component of the doublet exciton band [6]. Agreement between theory and experiment is difficult to evaluate because of the approximate nature of the calculation. One can only note that there is poorer agreement for NaCl than for

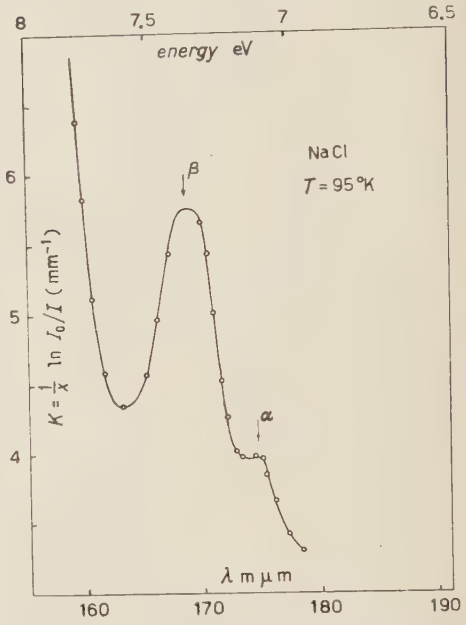

Fig. 2. – The absorption spectrum of a crystal of NaCl X-rayed at 95 °K for 2 hours (50 kV, 20 mA).

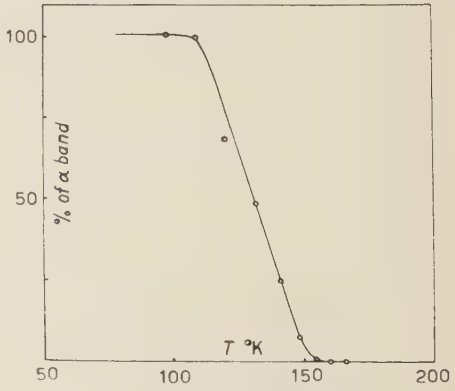

Fig. 3. – The annealing of the α band in a KCl crystal colored with X-rays at 95 °K.

[5] F. BASSANI and N. INCHAUSPÉ: *Phys. Rev.*, **105**, 819 (1957).

[6] J. E. EBY, K. J. TEEGARDEN and D. B. DUTTON: *Phys. Rev.*, **116**, 1099 (1959).

TABLE II. – *Position of the bands at liquid nitrogen temperature, comparison with calculated values* (eV).

	$h\nu_{free} - h\nu_\beta$		$h\nu_{free} - h\nu_\alpha$		$h\nu_\beta - h\nu_\alpha$	
	Th.	Exp.	Th.	Exp.	Th.	Exp.
KCl	.28	.31	.53	.76	.25	.45
KBr	.29	.35	.49	.63	.20	.28
KI	.23	.32	.44	.59	.21	.27
NaCl	.19	.59	.50	.85	.31	.26

$h\nu_{free}$ = energy of first exciton peak, experimental values from ref. [7].

Experimental values for $h\nu_\alpha$ and $h\nu_\beta$ from ref. [1,2], and the present work.

All theoretical values from ref. [5]. The experimental values for KBr given in ref. [5] differ from those above due to the recent measurement of $h\nu_{free}$.

the potassium salts and that the α bands occur at considerably lower energies than were predicted.

FUCHS [7] has computed the oscillator strength and autoionization lifetime of the β band in NaCl. He found that the β band should consist of six components, four of which are probably well into the fundamental absorption band and are unobservable. The other two should be separated by 0.10 eV, giving rise to a doublet β band. Since this splitting is a result of the spin-orbit coupling of the hole distributed on the Cl^- ions nearest the F-center, a similar result is expected in KCl. A splitting of the β band is not observed in the curves of Fig. 1 and 2, although a doublet is seen for exciton production in perfect crystals [6,8]. Measurements at lower temperatures may clarify this rather obscure point.

* * *

One of the authors (D.W.L.) wishes to express appreciation to the Commissione Americana per gli Scambi Culturali con l'Italia for a student Fulbright grant.

Note added in proof.

Since our work was sent to the publisher, R. ONAKA and I. FUJITA announced the publication in the *Phys. Rev.* of a paper on the same subject. (Abstract appeared in *Phys. Rev. Lett.*, **5**, 29 (1960)). We thank Prof. R. ONAKA for having sent us a copy of his work prior to publication.

(7) R. FUCHS: *Phys. Rev.*, **111**, 387 (1958).

(8) P. L. HARTMAN, J. R. NELSON and J. S. SIEGFRIED: *Phys. Rev.*, **105**, 123 (1957).

Effect of Lepton Non-Conservation on π-Decay.

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(*Nuovo Cimento*, **17**, 438 (1960))

The principal result (4) and conclusion of this paper are incorrect; the factors m_π^2/m_e^2 and m_π^2/m_μ^2 should not be present. For equal muon and electron coupling constants, the branching ratio is unaltered from the Ruderman-Finkelstein value, consequently no conclusion can be drawn about lepton conservation.

LIBRI RICEVUTI E RECENSIONI

P. ROMAN – *Theory of Elementary Particles*. North Holland Publishing Co. Amsterdam, pp. 575. Fior. 50.

Negli ultimi dieci anni le scoperte sperimentali nel campo delle particelle elementari e le speculazioni relative alle loro proprietà hanno fatto tali progressi da permettere di raccogliere in questo tempo relativamente breve un insieme di conoscenze sia fenomenologiche che teoriche così vasto e con carattere così spiccatamente proprio, da costituire un vero e proprio ramo a sè stante della fisica del microcosmo; nondimeno, l'avviamento alla conoscenza in tale dominio e lo studio sistematico di esso era finora reso particolarmente difficile ed arduo ad ogni giovane studioso che si fosse proposto di prepararsi esaurientemente per affrontarlo, dalla dispersione e dalla frammentarietà delle sorgenti di informazioni ad esso relative. Se, da un lato infatti, il panorama fenomenologico delle particelle gli poteva essere presentato in diversi articoli riassuntivi abbastanza esaurienti, le basi teoriche invece, per la comprensione e l'interpretazione delle loro proprietà, dovevano venire ricercate alle fonti più disparate: da un lato per la teoria generale delle particelle occorreva ricorrere a trattati generali di teoria dei campi, i quali sovente hanno un intento prevalentemente formale ed un interesse metodologico inteso a giustificare i procedimenti analitici usati in senso generale e non in relazione alle proprietà specifiche degli enti ai quali il formalismo si applica; dall'altro lato, per lo studio delle simmetrie inerenti alle interazioni bisognava appoggiarsi soprattutto a trattati classici di teoria dei gruppi, i quali anche quando sono scritti con intento prevalentemente matematico, si riferiscono quasi sempre ad esempi fisici tratti dall'atomo più che dal nucleo; sicchè in ultima analisi, l'unica fonte di informazioni sui problemi veramente vitali e direttamente connessi con la fisica delle particelle era conseguibile soltanto risalendo ai lavori originali, i quali sovente sono scritti al solo fine di essere intesi da altri specialisti e senza il minimo tentativo da parte degli autori di renderli accessibili ad una vasta cerchia di fisici.

L'opera teorica di P. Roman sulla fisica delle particelle elementari viene per prima, a notizia dello scrivente, a colmare nel modo più felice questa grave lacuna di fonti raggiungibili in tale dominio. Ed il suo pregio essenziale sta proprio nello spirito col quale essa è scritta, consistente nel prelevare da ogni singolo campo e nel presentare di esso solo quello che veramente interessa a chi lavori sulle particelle, lasciando a trattati più specializzati le difficili basi matematiche o le questioni di rigore; nel connettere continuamente ogni formalismo usato coll'aspetto specifico e

concreto delle particelle che esso traduce; e nell'esporre ogni concetto ed ogni procedimento con una precisione ed un dettaglio così spinti da renderli non solo comprensibili a fondo, ma adoperabili ed estensibili ad altri problemi analoghi; di modo che il lettore che ha attentamente seguito i vari passi dell'opera si troverà infine in possesso, sia di una visione generale armonica e coerente del formalismo interpretativo indispensabile alla comprensione della fisica delle particelle, che di uno strumento atto a renderlo capace di lavorare in modo produttivo in quel campo.

Il libro si apre con un capitolo destinato a fornire ai lettori le basi essenziali di carattere gruppale di cui necessiterà nel seguito. Ma in questa presentazione le nozioni generali della teoria astratta dei gruppi e dell'algebra degli operatori lineari come pure la teoria delle rappresentazioni sono ridotte al minimo necessario e servono solo da introduzione allo studio dettagliato delle trasformazioni ortogonali tetradimensionali e delle loro principali rappresentazioni partendo dal gruppo di trasformazioni complesse per giungere, attraverso il gruppo delle trasformazioni reali ed il gruppo di Lorentz, fino ai sottogruppi tridimensionali; in quanto proprio l'insieme di queste trasformazioni è la base essenziale sulla quale si innestano le simmetrie che interessano la fisica delle particelle, questa chiara presentazione è di notevole aiuto alla comprensione di tutte le proprietà che vengono esposte in seguito.

I due capitoli successivi sono dedicati ai fondamenti della teoria dei campi classica e quantistica limitati però alla presentazione di quegli aspetti che sono in diretta corrispondenza con le proprietà delle particelle che rappresentano. Vengono cioè dati e discussi i lagrangiani classici liberi e di interazione e le equazioni del moto per tutte le possibilità di spin e di parità di interesse fisico, con particolare rilievo, nel caso degli spinori, per le rappresentazioni di Weyl e di Majorana, così importanti nella formulazione delle interazioni deboli. Gli aspetti della quantizzazione di campi sono compendiati nella rappresentazione tramite i numeri di occupazione con messa in rilievo delle differenze essenziali che ciò comporta rispetto alla formulazione classica. Questi capitoli sono evidentemente intesi a fornire la base minima necessaria anche per potervi applicare le operazioni gruppali destinate a mettere in rilievo le simmetrie ed invarianze cui sono soggette le particelle.

A questi seguono infine i due capitoli fondamentali del libro che elencano e descrivono in dettaglio le leggi di conservazione con le proprietà ad esse legate. Il primo di questi è dedicato alle invarianze connesse con la parte spazio temporale delle equazioni del moto e delle interazioni, ed essenzialmente alle operazioni di parità, di inversione temporale, di coniugazione di carica, ed alle regole di selezione che da queste conseguono con le loro applicazioni agli esempi più noti. Segue la trattazione delle connessioni tra queste varie operazioni e delle simmetrie che risultano dalle loro combinazioni, con particolare riguardo alle conseguenze del teorema TCP e agli effetti della violazione di parità nelle interazioni deboli. L'ultimo capitolo raccoglie invece le proprietà relative alle invarianze nello spazio isobarico. Il formalismo dello spin isobarico viene prima sviluppato in modo dettagliato per le interazioni puramente nucleoniche e pioniche, e successivamente esteso alle particelle strane secondo lo schema di Gell-Mann e Nishijima. Dopo una presentazione piuttosto fenomenologica di questo, esso viene trattato in modo dettagliato secondo la teoria di D'Espagnat-Prentki. Infine, il formalismo dello spazio isobarico tetradimensionale viene svolto ed illustrato con l'esempio della teoria di Salam-Polkinghorne, a cui fa seguito un accenno a teorie più recenti sul tipo della teoria non lineare di Heisenberg. Se indubbiamente tale ultima parte è un po' manchevole dal punto di

vista dell'estensione, in quanto diversi interessanti tentativi di interpretazione delle proprietà isobariche delle interazioni forti non sono citati, non bisogna dimenticare che l'intento del libro è certamente più formativo che non semplicemente descrittivo o riassuntivo, con l'ovvio scopo di rendere familiari i formalismi generali delle invarianze, illustrandoli con esempi ben scelti e da tutti accettabili, piuttosto che fornire una raccolta dei vari tentativi interpretativi finora compiuti.

In base a questo breve esame si può forse osservare come per teoria delle particelle elementari, l'autore intenda più che altro lo studio delle loro proprietà gruppali e qualitative e non delle proprietà analitiche degli algoritmi che descrivono le loro interazioni. Tutte le conclusioni che oggi si sanno trarre dalle relazioni dispersive e dalle proprietà di analiticità delle ampiezze di interazione non sono comprese nella visuale di quest'opera. Ma per quanto riguarda lo studio delle simmetrie, il libro attuale presenta un carattere di completezza di chiarezza ed ordine nella esposizione, che crediamo non abbia precedenti in tale campo. Per cui non se ne può non raccomandare abbastanza la lettura accurata a chiunque voglia introdursi nello studio della fisica delle particelle e la sua eventuale adozione come testo di base per un corso propedeutico su tale argomento.

N. Dallaporta

J. M. Cork – *Radioactivité et Physique Nucléaire.* Trad. in francese da J. P. Bodet; Dunod, Paris, 1950; pp. XIII-406.

Si tratta della seconda edizione aggiornata di un'opera destinata ad esporre, come l'Autore nota nella prefazione, ciò che può servire ad iniziare alla fisica nucleare gli studenti dei primi corsi delle scuole superiori.

Il volume è suddiviso in quattordici capitoli, dedicati alla radioattività naturale, alla struttura del nucleo, ai metodi di produzione e rivelazione delle radiazioni, alle particelle elementari, alla radiazione cosmica, alla fissione nucleare; chiude il volume un'appendice con tabelle e dati numerici. Ciascun capitolo è corredato di esercizi da svolgere.

Il carattere alquanto enciclopedico e l'intenzione divulgativa con cui vengono presentati gli argomenti, impediscono che il libro assuma una sua fisionomia ben definita: sopprimendo le nozioni non necessarie e ponendo le necessarie in giusta luce, potrebbe riuscire un buon testo per i corsi di specializzazione dei nostri istituti tecnici industriali.

L'attualità degli argomenti e il notevole e ordinato numero di informazioni possono peraltro riuscire utili a tutti coloro che, per cultura o diletto, intendono interessarsi ai problemi della fisica nucleare senza soverchio impegno.

C. Manduchi